Lie Groups

Let G be a Lie group, i.e. a manifold and a group where the group operations are differentiable. Let $L_g : G \to G$ be left multiplication and let $R_g : G \to G$ be right multiplication by g. These are diffeomorphisms from G to G satisfying

$$L_g L_h = L_{gh}, \qquad R_g R_h = R_{hg}.$$

Let $C(g) = L_g R_{g^{-1}}$; then

$$C(g)(x) = gxg^{-1}$$

is conjugation by g and we have

$$C(g)C(h) = C(gh).$$

Since $C(e) = e$ and $C(g)$ is differentiable, its differential is a linear map on the tangent space T_eG. We denote $C(g)_* = L_{g*}R_{g^{-1}*}$ by $\mathrm{Ad}(g)$ and T_eG by $\mathfrak{g}$. Then $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is linear and satisfies $\mathrm{Ad}(g)\mathrm{Ad}(h) = \mathrm{Ad}(gh)$ for all g, h in G. The homomorphism $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ is the *adjoint action*.

If M is a manifold and $\phi : M \to M$ a diffeomorphism and X a vector field on M, we can define a vector field $\phi_*(X)$ by setting

$$\phi_*(X)(p) = \phi_*(X(\phi^{-1}(p))).$$

It follows then for any f that $\phi_*(X)(f) = (X(f \circ \phi)) \circ \phi^{-1}$ or $(\phi_*(X)(f)) \circ \phi = X(f \circ \phi)$. If X and Y are vector fields, this implies

$$\phi_*(X)(\phi_*(Y)(f)) \circ \phi = X((\phi_*(Y)(f)) \circ \phi) = X(Y(f \circ \phi)).$$

Switching X and Y and subtracting yields

$$\phi_*([X, Y]) = [\phi_* X, \phi_* Y].$$

As an application of this, we say a vector field X on a Lie group G is *left-invariant* if $L_{g*}X = X$ for all g. Then we conclude that *the bracket of left-invariant vector fields is also left-invariant.*

If ξ is a tangent vector at the identity $e \in G$, there is a unique left-invariant vector field X on G that equals ξ at e. This X is given by $X(g) = L_{g*}\xi$. If ξ, η are tangent vectors at e, then extend them to left-invariant vector fields X, Y on G; then $Z = [X, Y]$ is left-invariant, thus we can define a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by $(\xi, \eta) \mapsto Z(e)$. This is the *Lie bracket* on the tangent space $\mathfrak{g} = T_eG$ at e, and satifies

(1) $[\xi, \eta] = -[\eta, \xi]$,
(2) $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0$,
(3) $[\xi + \eta, \zeta] = [\xi, \zeta] + [\eta, \zeta]$, $a[\xi, \eta] = [a\xi, \eta]$.

These properties are an immediate consequence of the properties of bracketing of vector fields. These are the axioms for a Lie algebra, and thus $\mathfrak{g}$ is called *the Lie algebra of* G. Thus to each Lie group is associated a finite-dimensional Lie algebra, its tangent space $T_eG = \mathfrak{g}$ at e.

Let X be a left-invariant vector field on G, and let $\alpha(t, g)$ be the integral curve of X passing through g at $t = 0$; then

$$\frac{d}{dt} g\alpha(t, e) = \frac{d}{dt} L_g \alpha(t, e) = L_{g*} \frac{d}{dt} \alpha(t, e) = L_{g*} X(\alpha(t, e)) = X(g\alpha(t, e)).$$

By uniqueness, it follows that $\alpha(t,g) = g\alpha(t,e)$ whenever these curves are defined. Choosing $g = \alpha(s,e)$, we have

$$\alpha(s,e)\alpha(t,e) = \alpha(t,\alpha(s,e)) = \alpha(s+t,e).$$

If $\alpha(t,e)$ is defined on the interval $(-2\delta, 2\delta)$, then by defining $\alpha(t,e) = \alpha(\delta,e)\alpha(t-\delta,e)$, $\delta < t < 3\delta$, we extend $\alpha(t,e)$ to $(-2\delta, 3\delta)$. Continuing in this manner, we see $\alpha(t,e)$ is defined for all t. Thus the map $t \mapsto \alpha(t,e)$ is a group homomorphism from $\mathbf{R}$ into G, a *one-parameter subgroup.* If $\xi = X(e)$, we define $\exp(\xi) = \alpha(1,e)$.

To summarize,

$$\text{left-invariant vector fields} \quad \Longleftrightarrow \quad \mathfrak{g} = T_eG \quad \Longleftrightarrow \quad \text{one-parameter subgroups}$$

are in one-to-one correspondence, and we have defined a differentiable map $\exp : \mathfrak{g} \to G$ such that $\exp(t\xi)$, $t \in \mathbf{R}$, is the one-parameter subgroup associated with $\xi \in \mathfrak{g}$.

A vector field X on G is *right-invariant* if $R_{g*}X = X$ for all g. Starting with $\xi \in \mathfrak{g}$, we can construct the left-invariant vector-field $L_{g*}\xi$ and the right-invariant vector field $R_{g*}\xi$. When do we get the same vector-field?

LEMMA **1.** *A left-invariant vector field X is also right-invariant iff $\xi = X(e)$ satisfies*

$$Ad(g)(\xi) = \xi, \qquad g \in G,$$

i.e. ξ is a fixed point of the adjoint action.

Proof. To see this, suppose $X = L_{g*}X = R_{g*}X$. Evaluating at g yields $L_{g*}(\xi) = R_{g*}(\xi)$ hence

$$\xi = R_{g^{-1}*}L_{g*}(\xi) = \mathrm{Ad}(g)(\xi).$$

Conversely, suppose ξ is a fixed point and let $X(g) = L_{g*}(\xi)$ its left-invariant extension. Then

$$\begin{aligned} R_{g*}(X(p)) &= R_{g*}L_{p*}(\xi) = L_{p*}R_{g*}(\xi) \\ &= L_{p*}L_{g*}L_{g^{-1}*}R_{g*}(\xi) = L_{pg*}\mathrm{Ad}(g^{-1})(\xi) = L_{pg*}(\xi) = X(pg). \end{aligned} \tag{1}$$

Thus X is right-invariant. ∎

Now we look at invariant metrics. We say a metric $\langle \cdot, \cdot \rangle_p$ is *left-invariant* if

$$\langle L_{g*}v, L_{g*}w \rangle_{gp} = \langle v, w \rangle_p$$

for every $g \in G$ and $v, w \in T_pG$. This means the diffeomorphisms L_g are isometries. Similarly we define right-invariant metrics. A metric is *bi-invariant* if it is both left- and right-invariant.

When is a left-invariant metric bi-invariant? The following says we need only check Ad-invariance at the identity.

LEMMA **2.** *A left-invariant metric is also right-invariant iff*

$$\langle Ad(g)v, Ad(g)w \rangle_e = \langle v, w \rangle_e \tag{2}$$

for all $v, w \in \mathfrak{g}$.

Proof. If the metric is both left and right invariant, we have

$$\langle v, w \rangle_p = \langle R_{g*}v, R_{g*}w \rangle_{pg}$$

hence

$$\langle v, w \rangle_e = \langle L_{g*}v, L_{g*}w \rangle_g = \langle R_{g^{-1}*}L_{g*}v, R_{g^{-1}*}L_{g*}w \rangle_{gg^{-1}} = \langle \mathrm{Ad}(g)v, \mathrm{Ad}(g)w \rangle_e$$

which is (0.2) (Note R_g and L_h commute). Conversely, if (0.2) holds,

$$\langle v, w\rangle_g = \langle L_{g*}v, L_{g*}w\rangle_e = \langle \mathrm{Ad}(g^{-1})(L_{g*}v), \mathrm{Ad}(g^{-1})(L_{g*}w)\rangle_e = \langle R_{g*}v, R_{g*}w\rangle_e$$

so the metric also right invariant. □

Now let G be a compact Lie group, select an inner product on $\mathfrak{g}$, and extend it to a right invariant metric on G. Then the corresponding riemmanian measure μ is also right-invariant. Now define

$$\langle\langle v, w\rangle\rangle_e = \int_G \langle \mathrm{Ad}(g)v, \mathrm{Ad}(g)w\rangle_e \, d\mu(g).$$

Then using $\mathrm{Ad}(gh) = \mathrm{Ad}(g)\mathrm{Ad}(h)$, we see that $\langle\langle \cdot, \cdot\rangle\rangle_e$ satisfies (0.2), hence extends to a bi-invariant metric on G. Note we used compactness to guarantee that the integral is finite.

A *left Haar measure* is a left-invariant Borel measure on G, and a *right Haar measure* is a right-invariant Borel measure on G. We conclude the following:

Theorem **1.** *A compact Lie group has a bi-invariant reimannian metric, and consequently a bi-invariant Haar measure*

As a consequence, we also have

Theorem **2.** *A compact Lie group has a unique left Haar measure, up to a multiplicative constant, and a unique right Haar measure, up to a multiplicative constant, both equal the unique bi-invariant Haar measure.*

Proof. Let μ be the bi-invariant measure and let ν be right-invariant. Then multiply

$$\int_G f(gx)\, d\mu(x) = \int_G f(x)\, d\mu(x)$$

by $\phi(g)$ and integrate to get

$$\int_G \int_G \phi(g) f(gx)\, d\mu(x) d\nu(g) = \left(\int_G \phi(g)\, d\nu(g)\right)\left(\int_G f(x)\, d\mu(x)\right).$$

Switch the order of integration and replace g bygx^{-1}; since ν is right-invariant, we obtain

$$\int_G \int_G \phi(gx^{-1}) f(g)\, d\nu(g) d\mu(x) = \left(\int_G \phi(g)\, d\nu(g)\right)\left(\int_G f(x)\, d\mu(x)\right).$$

Switch the order back and replace x by xg; since μ is right-invariant, we obtain

$$\int_G \int_G \phi(x^{-1}) f(g)\, d\mu(x) d\nu(g) = \left(\int_G \phi(g)\, d\nu(g)\right)\left(\int_G f(x)\, d\mu(x)\right)$$

or

$$\left(\int_G \phi(x^{-1})\, d\mu(x)\right)\left(\int_G f(g)\, d\nu(g)\right) = \left(\int_G \phi(g)\, d\nu(g)\right)\left(\int_G f(x)\, d\mu(x)\right).$$

Since ϕ a fixed function, this shows μ and ν are equal up to a multiplicative constant. □