On some degenerate deformations of commutative polynomial algebras

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Abstract

The prime ideals, primitive ideals, and irreducible representations of $S = K\{x_1, \ldots, x_n\}/\langle x_i x_j = \alpha_{ij}, i < j\rangle$, for K an algebraically closed field, $n \geq 3$, and $\alpha_{ij} \in K$, are classified. This classification is accomplished by reducing the n-variable case to the study of $R = K\{x, y, z\}/\langle xy = 0, yz = 0, xz = 1\rangle$. The finite-dimensional irreducible representations of R are classified using a result of Adam Berliner, which is included as an appendix. The classification of the prime ideals of R proves that, under the Zariski topology, the topological dimension of S is no greater than the Gelfand-Kirillov dimension of R.

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1 Introduction

In the 1930's, Jacobson began to explore cases of algebras in non-commuting variables over a field, modulo an ideal generated by relations of degree less than or equal to two. In particular, Jacobson was interested in the prime and primitive spectra of these algebras. Algebras of this type are particularly interesting because, as filtered vector spaces, they are very close to commutative algebras. Thus these algebras can be viewed as natural generalizations of commutative polynomial algebras and as being related to quantum groups.

Since Jacobson studied algebras of this type, there have been numerous other successful studies of the prime ideals and representations of iterated Ore extensions and of other finitely generated noetherian algebras (e.g., quantum groups, enveloping algebras, noetherian group algebras). Irving, in the 1970's, studied the prime ideal structure of arbitrary Ore extensions of a commutative noetherian ring (see [7] and [8]). Gerritzen also played an important role by classifying the irreducible representations of $k\{x, y\}/\langle yx - 1\rangle$, where k is an algebraically closed field (see [1]).

Until now, the algebras considered have primarily been noetherian domains. In this paper, we are concerned with deformations of these algebras. In particular, this paper will give a classification of the prime and primitive ideals of

$$K\{x_1, \cdots, x_n\}/\langle x_i x_j = 0 x_j x_i + \alpha_{ij}, i < j\rangle = K\{x_1, \cdots, x_n\}/\langle x_i x_j = \alpha_{ij}, i < j\rangle,$$

where K is an algebraically closed field, $n \geq 3$, and $\alpha_{ij} \in K$. The prime and primitive spectra of some classes can be immediately classified. For the remaining cases, we reduce our study of the prime and primitive ideal structure of the n-variable case to the study of the prime and primitive ideals of $R = K\{x, y, z\}/\langle xy = 0, yz = 0, xz = 1\rangle$.

Well-known work has shown that R has many "bad" properties. For instance, R has non-Goldie prime factors and infinite Krull dimension. However, the Gelfand-Kirillov dimension of R is 3. Furthermore, the classification of the prime ideals of R proves the following corollary, showing that the prime spectrum of R is fairly well-behaved.

Corollary 1.1. *Under the Zariski topology, the topological dimension of R is no greater than the Gelfand-Kirillov dimension of R.*

After reducing our study of algebras in n-variables to the study of R, we prove some preliminary results about this algebra. In particular, we note that a more general result given in the appendix allows us to completely classify the finite dimensional irreducible representations of R. Next, we explicitly construct an infinite family of infinite dimensional irreducible representations, using a method of Irving. A proof is then given showing that the representations already classified are indeed all of the irreducible representations of R. Finally, we classify all of the prime ideals of R.

2 Reduction to Three Variable Cases

Throughout this paper, we will use x, y, and z to stand for their images in various quotient algebras. We will now show that to complete a classification of the prime and primitive spectra of algebras of the form $K\{x_1, \cdots, x_n\}/\langle x_i x_j = \alpha_{ij}, i < j\rangle$, where K is an algebraically closed field, $n \geq 3$, and $\alpha_{ij} \in K$, we must classify the prime and primitive ideals of $R = K\{x, y, z\}/\langle xy = 0, yz = 0, xz = \gamma\rangle$ where γ is a nonzero element of K. Throughout, let $n \geq 3$.

Let $S_1 = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j = 1, i < j\rangle$. By manipulating the relations on the variables $x_1, x_2, \cdots, x_n$, one sees that $S_1 \cong K[x]/\langle x^2 - 1\rangle$. Hence all the prime and primitive ideals of S_1 are known.

Next, let $S_2 = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j = 0, i < j\rangle$. Note that $x_j x_i$ is contained in the nilradical of S_2 for $1 \leq i < j \leq n$. Thus, modulo its nilradical, S_2 is commutative and hence the classification of the prime and primitive ideals is clear.

Let $S_3 = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j = \alpha_{ij}, i < j\rangle$ where the $\alpha_{ij} \in K$, $n \in \mathbb{N} > 3$, two or more of the $\alpha_{ij} \neq 0$, and one or more of the $\alpha_{ij} = 0$. By manipulating the relations, the variables equal 0 and hence, $S_3 = 0$.

Finally, let $S_4 = K\{x_1, x_2, \cdots, x_n\}/\langle x_i x_j = \alpha_{ij}, i < j\rangle$ with $\alpha_{ij} \in K$. Further suppose that there exists a k and an ℓ such that $\alpha_{k\ell} = 1$ and $\alpha_{ij} = 0$ for $i \neq k$ and $j \neq \ell$.

Proposition 2.1. *The algebra S_4 (as above) is trivial except when $k = 1$ and $\ell = n$.*

Proof. Suppose $\alpha_{k\ell} = 1$ and $\alpha_{ij} = 0$ for $i \neq k$ and $j \neq \ell$. If there exists an m such that $1 \leq m < k$, then $x_\ell x_k = 1$ which implies $x_m(x_\ell x_k) = x_m$. This implies that $(x_m x_\ell)x_k = x_m$ and thus that $x_m = 0$. Proceed similarly if $\ell < n$. □

Proposition 2.2. *Let S_4 be as above. Let P be a prime ideal of S_4. Then at most one of $x_2, \cdots, x_{n-1}$ is not contained in P.*

Proof. Let P be any prime ideal of S_4. Suppose there exists a $2 \leq k \leq n-1$ such that $x_k \notin P$. Elements of $\langle x_\ell \rangle \langle x_k \rangle$ are sums of elements of the form $r_1 x_\ell r_2 x_k r_3$ where $r_1, r_2, r_3 \in S_4$. If $k \neq 2$, then, for any $2 \leq \ell < k$, $x_\ell r_2 x_k = 0$. Hence $\langle x_\ell \rangle \langle x_k \rangle = 0$. Furthermore, since P is prime and $x_k \notin P$, the element $x_\ell \in P$. Similarly, if $x_k \neq n-1$, then for any $k < m \leq n-1$, the element $x_m \in P$. Hence, if $x_k \notin P$ for $2 \leq k \leq n-1$, then $x_\ell \in P$ for all ℓ such that $2 \leq \ell \leq n-1$ and $\ell \neq k$. Therefore any prime ideal P of S_4 will contain all of the $x_2, \cdots, x_{n-1}$ or all but one of the $x_2, \cdots, x_{n-1}$. □

Hence, using the classification of the prime and primitive ideals of $K\{x, y, \}/\langle xy - 1 \rangle$ (see [8]), a classification of the prime and primitive ideals of R, will completely determine the prime and primitive spectra of S_4. Furthermore, using Gerritzen's classification of the irreducible representations of $K\{x, y, \}/\langle xy - 1 \rangle$ (see [1]), a classification of the irreducible representations of R will complete such a classification for S_4.

3 Prime and Primitive Spectra of R

3.1 Notation and Preliminary Results

For the remainder of this work, whenever we say that a is a factor of b, for elements a and b in a ring, we mean $b = ras$ for some elements r and s of this ring. Also, $K\langle r_1, \cdots r_i \rangle$, where $r_1, \cdots, r_i \in R$, will refer to the K-subalgebra of R generated by $r_1, \cdots, r_i$. We will identify the K-subalgebra of R generated by y, $K\langle y \rangle$, with the K-algebra $K[y]$.

Proposition 3.1. *The set* $\{z^i y^j x^k : i,\ j,\ k$ *are non-negative integers*$\}$ *is a* K*-linear basis for* R.

Proof. Let $A = K[x][y; \sigma_1]$ be a right Ore extension of $K[x]$ with $\sigma_1(x) = 0$. Then, by 1.2.3 of [16], the set $\{y^j x^k : j$ and k are non-negative integers$\}$ is a K-linear basis for A. Then $R = A[z; \sigma_2]$ is a right Ore extension of A with $\sigma_2(y) = 0$ and $\sigma_2(x) = \gamma$. Thus, by by 1.2.3 of [16], $\{z^i y^j x^k : i,\ j,\ k$ are non-negative integers$\}$ is a K-linear basis for R. □

Hence, whenever we refer to a nonzero element $\sum_{r=1}^{m} \alpha_r z^{i_r} y^{j_r} x^{k_r}$ in R, we assume that the (i_r, j_r, k_r) are distinct for distinct r and that the α_r are all nonzero.

Proposition 3.2. *Any nonzero ideal of R contains a nonzero element of $K[y]$.*

Proof. Let I be a nonzero ideal of R and $f = \sum_{r=0}^{n} \lambda_r z^{i_r} y^{j_r} x^{k_r}$ a nonzero element of I.

Case 1: Suppose there exists a summand s_1 of f with x not a factor of s_1. Then fy will be nonzero and x will not be a factor of any summand of fy. If there is at least one summand, s_2, of fy with z not a factor of s_2, then yfy will be a nonzero element of I without x or z as a factor of any of its summands.

Suppose then that fy has z as a factor of every summand. Let α be the y-degree of fy. Note that $\alpha \geq 1$ (since we have multiplied through by y). Let β be the highest z-degree of the summands of fy with y-degree α. Note that there can be only one summand of fy with y-degree α and z-degree β. Also, note that $x^\beta fy$ contains a summand of the form λy^α, for some $\lambda \in K$, and this summand cannot possibly cancel with any other summands (using the fact that there is only one summand of fy with y-degree α and z-degree β). Therefore, $x^\beta fy$ is nonzero.

Also, $x^\beta fy$ is an element of I without x as a factor of any of its summands and there exists at least one summand of $x^\beta fy$ without z as a factor. Hence, as previously, $yx^\beta fy$ will be a nonzero element of I that is an element without x or z as a factor of any of its summands. Therefore, the desired result is true in this case.

Case 2: Suppose there exists a summand of f without z as a factor. Then proceed similarly to the above case.

Case 3: Suppose every summand of f has x and z as a factor. Let α be the minimum exponent of x appearing in f. Note that $\alpha \geq 1$. Then fz^α cannot possibly be zero and fz^α will be an element of I where not every summand has x as a factor. Hence proceed as in Case 1.

Therefore, every nonzero ideal of R contains a nonzero element of $K[y]$. □

Corollary 3.3. *R is prime.*

Proof. Suppose I and J are nonzero ideals of R with $IJ = 0$. By Proposition 3.2, I contains a nonzero polynomial in y, say f, and J contains a nonzero polynomial in y, say g. Then $fg \in IJ$ and $fg \neq 0$ (the product of two nonzero polynomials in y cannot be zero), which is a contradiction. Hence, I or J must be zero.

□

Lemma 3.4. *In R, $\langle y^i \rangle = \langle y \rangle^i$, where i is any integer greater than or equal to zero.*

Proof. The proof is shown for $i = 2$ and the general case follows analogously. It is obvious that $\langle y^2 \rangle \subseteq \langle y \rangle^2$. Suppose $a \in \langle y \rangle^2$. Then a is a finite sum of elements of the form: $r_1 y z^i y^j x^k y r_2$ where r_1 and r_2, are elements of R and i, j, and k are nonnegative integers. Then, if either i or k is nonzero, $r_1 y z^i y^j x^k y r_2 = 0 \in \langle y^2 \rangle$. If $i = 0$ and $k = 0$, then $r_1 y z^i y^j x^k y r_2 = r_1 y^{j+2} r_2 \in \langle y^2 \rangle$.

□

Lemma 3.5. *Let P be any nonzero proper prime ideal of R not containing y. Then there exists a nonzero $c \in K$ such that $\langle y^2 - cy \rangle \subseteq P$.*

Proof. If we can prove that $\langle y^2 - cy \rangle \subseteq P$ for some $c \in K$, then, by Lemma 3.4, c is nonzero.

By Proposition 3.2, P contains a nonzero polynomial in y, say g. If g has a nonzero constant term, then xgz is equal to a nonzero scalar and $xgz \in P$, a contradiction. Hence, g must have a zero constant term, i.e., $g = yf$ for some polynomial $f \in K[y]$.

Suppose then that $f = (y - \alpha_1)(y - \alpha_2) \cdots (y - \alpha_t)$ is a factorization of f into irreducible polynomials over K. Since $\langle g(y) \rangle \subseteq P$ and P is prime, we need only prove

$$\langle y(y - \alpha_1) \rangle \langle y(y - \alpha_2) \rangle ... \langle y(y - \alpha_t) \rangle \subseteq \langle g(y) \rangle$$

to prove the lemma. We proceed by induction on t.

If $t = 2$, let $a \in \langle y(y - \alpha_1) \rangle \langle y(y - \alpha_2) \rangle$. Then a is equal to a finite sum of elements of the form $r_1 y(y - \alpha_1) z^i y^j x^k y(y - \alpha_2) r_2$, for some elements r_1 and $r_2 \in R$ and nonnegative integers i, j, and k. If i or k is nonzero, then

$$r_1 y(y - \alpha_1) z^i y^j x^k y(y - \alpha_2) r_2 = 0 \in \langle y(y - \alpha_1)(y - \alpha_2) \rangle.$$

If i and k are zero, then

$$r_1 y(y - \alpha_1) z^i y^j x^k y(y - \alpha_2) r_2 = r_1 y^{j+2} (y - \alpha_1)(y - \alpha_2) \in \langle y(y - \alpha_1)(y - \alpha_2) \rangle.$$

Hence, $a \in \langle y(y - \alpha_1)(y - \alpha_2) \rangle$ and the desired result is true for $t = 2$.

Suppose the claim is true for $t < s$ and suppose now that $t = s$. Using the induction hypothesis,

$$\langle y(y - \alpha_1) \rangle \langle y(y - \alpha_2) \rangle \ldots \langle y(y - \alpha_{t-1}) \rangle \subseteq \langle y(y - \alpha_1) \cdots (y - \alpha_{t-1}) \rangle.$$

Hence,

$$\langle y(y - \alpha_1) \rangle \langle y(y - \alpha_2) \rangle \cdots \langle y(y - \alpha_t) \rangle \subseteq \langle y(y - \alpha_1) \ldots (y - \alpha_{t-1}) \rangle \langle y(y - \alpha_t) \rangle.$$

Suppose

$$a \in \langle y(y-\alpha_1)\dots(y-\alpha_{t-1})\rangle\langle y(y-\alpha_t)\rangle.$$

Then a is a finite sum of elements of the form:

$$r_1(y-\alpha_1)\cdots(y-\alpha_{t-1})y(z^iy^jx^k)y(y-\alpha_t)r_2,$$

for elements r_1 and $r_2 \in R$ and nonnegative integers i, j, and k. Then, as in the case where $t=2$, if i or k is nonzero,

$$r_1(y-\alpha_1)\cdots(y-\alpha_{t-1})y(z^iy^jx^k))y(y-\alpha_t)r_2 = 0.$$

If i and k are zero, then

$$r_1(y-\alpha_1)\cdots(y-\alpha_{t-1})y(z^iy^jx^k))y(y-\alpha_t)r_2 = r_1y^{j+2}(y-\alpha_1)\cdots(y-\alpha_{t-1})(y-\alpha_t)$$
$$\in \langle y(y-\alpha_1)\cdots(y-\alpha_t)\rangle \subseteq P.$$

Hence, $a \in P$. This implies that

$$\langle y(y-\alpha_1)\rangle\langle y(y-\alpha_2)\rangle\cdots\langle y(y-\alpha_t)\rangle \subseteq P.$$

Hence, since P is prime, there exists an $1 \le i \le t$ such that $\langle y(y-\alpha_i)\rangle \subseteq P$, as desired. □

Note that $\langle y\rangle$ is a prime and primitive ideal of R (see [8]). Also note that as a corollary to Proposition .12 (see the appendix), the following is true.

Corollary 3.6. *All cofinite dimensional primitive ideals of R are of the form: $\langle y, x-\lambda, z-\lambda^{-1}\rangle$, where λ is a nonzero element of K.*

3.2 Infinite dimensional irreducible representations of R

To show the existence of infinite dimensional irreducible representations of R, we begin by explicitly constructing such a representation, using a method of [8].

3.2.1 Construction of an Infinite Dimensional Irreducible Representation of R

Let λ be a nonzero element of K and let M_λ be the infinite dimensional K-vector space with basis v_0, v_1, v_2,.... By the following action, M_λ is a $K\{x,y,z\}$-module:

(i) $zv_n = v_{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 0$,
(ii) $xv_0 = 0$
(iii) $xv_n = v_{n-1}$ for all $n \in \mathbb{N}$ with $n \geq 1$,
(iv) $yv_0 = \lambda v_0$, and
(v) $yv_n = 0$ for all $n \in \mathbb{N}$ with $n \geq 1$.

Let $v \in M_\lambda$. Since v_0, v_1, v_2,... is a K-linear basis for M_λ, we can write v in the form $v = \sum_{i=0}^{\ell} c_i v_i$, for some $c_i \in K$ and some nonnegative integer ℓ. Whenever we write an element of M_λ in this form, we are assuming that $c_\ell \neq 0$ and that the v_i are the K-basis vectors of M_λ. Therefore,

$$xy \cdot v = xy \cdot \sum_{i=0}^{\ell} c_i v_i = x \cdot c_0 \lambda v_0 = 0.$$

Hence, $xy \in ann_{K\{x,y,z\}}(M_\lambda)$. Similarly, one easily checks that yz and $xz - 1$ are elements of $ann_{K\{x,y,z\}}(M_\lambda)$. Hence, M_λ is an R-module.

Let $v = \sum_{i=0}^{\ell} c_i v_i$, (where $c_i \in K$ and v_i are K-basis vectors of M_λ), be an arbitrary element of M_λ with $c_\ell \neq 0$. Then $z^m x^\ell v = c_\ell v_m$, which implies that $Rv = M_\lambda$ or, equivalently since v is arbitrary, that M_λ is simple.

Proposition 3.7. *The ideal* $ann_R(M_\lambda) = \langle \lambda zx + y - \lambda \rangle$.

Proof. The proof is shown for $\lambda = 1$ and follows analogously for other values of λ. An easy check shows that $\langle zx + y - 1\rangle \subseteq ann_R(M_\lambda)$. Let $R' = R/\langle zx + y - 1\rangle$ and let x, y, and z denote their images in R'. Then $\{z^i y^j x^k : i,\, j,\, k$ are integers$\}$ spans R' over K. However, $y^2 - y \in \langle zx + y - 1\rangle$ and hence $y^2 - y = 0$ in R'. Therefore, $\{z^i y^j x^k : i,\, j$ and k are integers with $j = 0$ or $1\}$ spans R' over K. Also, $zx = 1 - y$ in R'. Thus, $\{z^i y x^k : i$ and k are integers$\} \bigcup \{z^i : i$ is an integer$\} \bigcup \{x^k : k$ is an integer$\}$ spans R' over K. Hence, if r is any nonzero element of R', we will write r in the form:

$$r = \sum_a \lambda_a z^{i_a} y x^{k_a} + \sum_b \gamma_b z^b + \sum_c \mu_c x^c,$$

for nonzero λ_a, γ_b, and $\mu_c \in K$ and for (i_a, k_a) distinct for distinct a.

Suppose

$$0 \neq p = \sum_a \lambda_a z^{i_a} y x^{k_a} + \sum_b \gamma_b z^b + \sum_c \mu_c x^c \in ann_{R'}(M_\lambda),$$

where the λ_a, γ_b, and $\mu_c \in K \setminus \{0\}$. Let q be greater than k_a for all a and c for all c. Then

$$pv_q = \sum \gamma_b v_{q+b} + \sum \mu_c v_{q-c},$$

which must be zero since $p \in ann_R(M_\lambda)$.

This implies that for each b, there exists a c such that $\gamma_b v_{n+b} = -\mu_c v_{n-c}$. This, in turn, implies that $n+b = n-c$ which can only happen if $b = c = 0$ since $b, c \geq 0$. Similarly for each c, there exists a b such that $\gamma_b v_{n+b} = -\mu_c v_{n-c}$. Therefore, $c = b = 0$. Hence, $p = \sum_a \lambda_a z^{j_a} y x^{k_a}$.

Let r the highest exponent of x appearing in p. Then

$$pv_r = \sum_\ell \lambda_\ell z^{i_\ell} y x^{k_\ell} v_r = \sum_{(\ell:k_\ell=r)} \lambda_\ell z^{i_\ell} y v_0 = \sum_{(\ell:k_\ell=r)} \lambda_\ell z^{i_\ell} v_0 = \sum_{(\ell:k_\ell=r)} \lambda_\ell v_{i_\ell}.$$

However, for each ℓ such that $k_\ell = r$, the i_ℓ must be distinct (otherwise there would be two summands with the same x, y, and z-degrees). This implies that $\sum_{(\ell:k_\ell=r)} \lambda_\ell v_{i_\ell}$ cannot possibly be zero, which contradicts that $p \in ann_R(M_\lambda)$. Hence, $0 = ann_{R'}(M_\lambda)$, which implies that $\langle zx + y - 1\rangle \supseteq ann_R(M_\lambda)$ and, hence, that $ann_R(M_\lambda) = \langle zx + y - 1\rangle$, as desired.

□

Thus, R has an infinite family of infinite dimensional irreducible representations. We now wish to explore the existence of infinite dimensional irreducible representations of R that are not isomorphic to those already classified.

3.2.2 Classification of Irreducible Infinite Dimensional Representations of R

For a classification of the irreducible infinite dimensional representations V of R with $y \in ann_R V$, see [1]. We now concentrate on irreducible infinite dimensional representations of R without y in their annihilator.

Proposition 3.8. *Any nonzero infinite dimensional simple R-module M with $ann_R(M) \neq 0$ and $y \notin ann_R(M)$ is isomorphic to M_λ (as defined above) for some $\lambda \neq 0 \in K$.*

Proof. Let M be any nonzero infinite dimensional simple R-module. Suppose $ann_R(M) \neq 0$. Let $A = K\langle x, y\rangle$ and $M' = (AyxA)M$.

Case 1: Suppose $M' = 0$. This implies that yx is in the annihilator in R of M. Hence $yxz = y \in ann_R(M)$, contradicting the assumption that $y \notin ann_R(M)$.

Case 2: Suppose $M' \neq 0$. Then $x \in ann_A(M')$. Therefore, M' is a $K[y]$-module. By Proposition 3.2, there exists a $f(y) \in ann_R(M)$ and, hence, $f(y)M' = 0$. Therefore, for $0 \neq m \in M'$, Am is finite dimensional over K. Thus Am contains a simple A-module, say Av_0 for some $v_0 \in M'$. Thus $xv_0 = 0$ and $yv_0 = \lambda v_0$ for some $\lambda \neq 0 \in K$.

Choose a basis for M over K that includes v_0, say $v_0, v_1, v_2, \ldots$. Since $xzv_i = v_i$ for all i, $v_0 \notin zM$. Suppose n is any positive integer such that $v_n \notin zM$. Since M is a simple R-module, there exists an element, $\sum_{\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell}$, of R such that

$$\left(\sum_{\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell}\right) v_0 = v_n.$$

Since $xv_0 = 0$, $k_\ell = 0$ for all ℓ. Using the fact that $yv_0 = \lambda v_0$,

$$\sum_{\ell=0}^{m} \alpha_\ell \lambda^{j_\ell} z^{i_\ell} \cdot v_0 = v_n.$$

Then

$$x^m \left(\sum_{\ell=0}^{m} \alpha_\ell \lambda^{j_\ell} z^{i_\ell}\right) \cdot v_0 = \alpha_m \lambda^{j_m} v_0$$

and

$$x^m \left(\sum_{\ell=0}^{m} \alpha_\ell \lambda^{j_\ell} v_0\right) = x^m \cdot v_n.$$

Hence, $\alpha_m \lambda^{j_m} v_0 = x^m \cdot v_n$.

However, we also have from the relations in R that $x^m z^m v_0 = v_0$. Thus, $x^m z^m v_0 = (\alpha_m \lambda^{j_m})^{-1} x^m v_n$. This implies that

$$x^m \in ann_R(z^m v_0 - (\alpha_m \lambda^{j_m})^{-1} v_n),$$

which implies that

$$1 \in ann_R(z^m v_0 - (\alpha_m \lambda^{j_m})^{-1} v_n).$$

Hence, $z^m v_0 = (\alpha_m \lambda^{j_m})^{-1} v_n$, contradicting that $v_n \notin zM$. Therefore, v_0 is the only basis vector of M not in zM.

Therefore, we have the following:

(i) for all $n > 0, x \cdot v_n \neq 0$ since $xz \cdot v_i = v_i$ for all i,

(ii) for all $n > 0, y \cdot v_n = 0$ since $yz \cdot v_i = 0$ for all i, and, hence,

(iii) $y \cdot v_0 = \lambda v_0, y \cdot v_i = 0$ for all $i > 0$,

(iv) $x \cdot v_0 = 0$,

(v) $x \cdot v_i \neq 0$ for all $i > 0$, and

(vi) if $z \cdot v_i = v_j$, then $x \cdot v_j = v_i$.

By the above argument, for any basis vector, v_i, $v_i = z^a \cdot v_0$ for some nonnegative integer a.

Next, let $v_0 = w_0, z \cdot v_0 = w_1, z^2 \cdot v_0 = w_2, z^3 \cdot v_0 = w_3, \ldots$. Note that this list will include all basis vectors of M over K and hence the basis $w_0, w_1, \ldots$ is just a renumbering of the original basis, $v_0, v_1, \ldots$. Let $u_0, u_1, \ldots$ be the basis for M_λ (M_λ as defined above), with R-action:

(i) $zu_n = u_{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 0$,

(ii) $xu_0 = 0$,

(iii) $xu_n = u_{n-1}$ for all $n \in \mathbb{N}$ with $n \geq 1$,

(iv) $yu_0 = u_0$, and

(v) $yu_n = 0$ for all $n \in \mathbb{N}$ with $n \geq 1$.

Then, define $\phi : M \to M_\lambda$ by $\phi(w_i) = u_i$. This is easily seen to be an isomorphism of R-modules. □

Thus, all nonzero infinite dimensional irreducible representations of R have been classified. To complete a classification of the primitive ideals of R, we need to ascertain whether or not $\langle 0 \rangle$ is a primitive ideal.

Proposition 3.9. *R is not primitive.*

Proof. Let M be a faithful, simple R-module. Let $A = K\langle x, y\rangle$ and $M' = (AyxA)M$, as above. Note that $M' \neq 0$ since M is faithful. Thus, for all $v \in M'$ and for all nonzero $f(y) \in K[y]$, $f(y) \cdot v \neq 0$, or else $M = M_\lambda$ for some $\lambda \in K$ as in the above proof.

Choose $v \neq 0 \in M'$. Then, since M is simple, $Ryv = M$. This implies that there exists an $a \in R$ such that $ayv = v$, say $a = \sum_{\ell=0}^m \alpha_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell}$, for some scalars α_ℓ.

Then

$$\left(\sum_{\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell} x^{k_\ell}\right) y \cdot v = \left(\sum_{\ell:k_\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell+1}\right) \cdot v.$$

Let α be the minimum exponent of z appearing in any summand of

$$\left(\sum_{\ell:k_\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell+1}\right).$$

Then

$$x^\alpha \left(\sum_{\ell:k_\ell=0}^{m} \alpha_\ell z^{i_\ell} y^{j_\ell+1}\right) \cdot v = 0,$$

since $x \in ann_R(M')$. This implies that

$$\left(\sum_{\ell : k_\ell = 0, i_\ell > \alpha} z^{i_\ell - \alpha} y^{j_\ell + 1} + h(y) \right) \cdot v = 0,$$

where $h(y) \in K[y]$ and $deg(h(y)) \geq 1$. Hence,

$$y\left(\sum_{\ell : k_\ell = 0, i_\ell > \alpha} z^{i_\ell - \alpha} y^{j_\ell + 1} + h(y) \right) \cdot v = 0.$$

But this implies that $yh(y) \cdot v = 0$, contradicting that, for all nonzero $f(y) \in K[y]$, $f(y) \cdot v \neq 0$. □

3.3 Classification of the Prime Ideals of R

Proposition 3.10. *Any nonzero prime of R not containing y contains $\langle \lambda zx - \lambda + y \rangle$ for some nonzero $\lambda \in K$.*

Proof. Let P be a prime ideal of R not containing y. By Lemma 3.5, there exists a $\lambda \in K$ such that $y^2 - \lambda y \in P$. If $\langle \lambda zx - \lambda + y \rangle \langle y \rangle \subseteq P$, then because P is prime and P does not contain y, $\langle \lambda zx - \lambda + y \rangle \subseteq P$. Let $a \in \langle \lambda zx - \lambda + y \rangle \langle y \rangle$. Then a is equal to a finite sum of elements of the form:

$$r_1(\lambda zx - \lambda + y)(z^i y^j x^k)(y) r_2,$$

for some elements r_1 and $r_2 \in R$ and some nonnegative integers i, j, and k. Thus, if $k > 0$, then

$$r_1(\lambda zx - \lambda + y)(z^i y^j x^k)(y) r_2 = 0 \in P.$$

If $k = 0$ and $i > 0$, then

$$r_1(\lambda zx - \lambda + y)(z^i y^j x^k) y r_2 = r_1(\lambda zx - \lambda + y)(z^i y^{j+1}) r_2$$

$$= r_1(\lambda z^i yj + 1 - \lambda z^i yj + 1 + yz^i yj + 1) r_2 = 0 \in P.$$

If $k = 0$ and $i = 0$, then

$$r_1(\lambda zx - \lambda + y)(z^i y^j x^k) y r_2 = r_1(\lambda zx - \lambda + y)(y^{j+1}) r_2$$

$$= r_1(\lambda zxy^{j+1} - \lambda y^{j+1} + y^{j+2}) r_2 = r_1(-\lambda y^{j+1} + y^{j+2}) r_2 \in \langle y^2 - \lambda y \rangle \subseteq P.$$

Hence, $a \subseteq P$. □

Thus a classification of the prime ideals of $R/\langle \lambda zx - \lambda - y\rangle$ for all $\lambda \in K$ will complete the classification of the prime ideals of R.

Proposition 3.11. *All nonzero ideals of R properly containing $\langle \lambda zx - \lambda + y\rangle$ contain y.*

Proof. Let P be a nonzero ideal of $R/\langle \lambda zx - \lambda + y\rangle$. Suppose $a \neq 0 \in P$. Then, as in the proof of Proposition 3.7, a can be written in the following form:

$$a = \sum_{\ell_1=0}^{m_1} \alpha_{\ell_1} z^{i_{\ell_1}} y x^{k_{\ell_1}} + \sum_{\ell_2=0}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y + \sum_{\ell_3=0}^{m_3} \alpha_{\ell_3} y x^{k_{\ell_3}} + \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} z^{i_{\ell_4}} + \sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{k_{\ell_5}} + \lambda_1 y + \lambda_2,$$

with $\ell_n, k_n > 0$ for all $1 \leq n \leq 5$, $\alpha_n \in K$ for all $1 \leq n \leq 5$, and $\lambda_1, \lambda_2 \in K$.

Case 1: Suppose $\lambda_1 \neq 0$ and $\lambda_2 = 0$. Then $yay = \lambda_1 y \in P$, using the fact that $\lambda_1 y^2 = \lambda_1 y$ in $R/\langle \lambda zx - \lambda - y\rangle$.

Case 2: Suppose $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then $yay = \lambda_2 y \in P$.

Case 3: Suppose $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Let γ be strictly greater than the highest exponent of z appearing in a. Then:

$$x^\gamma a = \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} x^{\gamma - i_{\ell_4}} + \sum_{\ell_5=0}^{m_5} x^{\gamma + k_{\ell_5}} + \lambda_2 x^\gamma.$$

Therefore,

$$x^\gamma a z^\gamma = \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} z^{i_{\ell_4}} + \sum_{\ell_5=0}^{m_5} x^{k_{\ell_5}} + \lambda_2.$$

Thus, $y x^\gamma a z^\gamma y = \lambda_2 \in P$.

Case 4: Suppose $\lambda_1 = 0$ and $\lambda_2 = 0$.

Subcase a: Suppose $\sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{k_{\ell_5}} \neq 0$. Let γ be as in Case 2. Then

$$x^\gamma a = \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} x^{\gamma - i_{\ell_4}} + \sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{\gamma + k_{\ell_5}}.$$

Let δ be the minimum exponent of x appearing in $x^\gamma a$. Then

$$z^\delta x^\gamma a = \sum_{\ell_4 : \gamma - i_{\ell_4} > \delta}^{m_4} \alpha_{\ell_4} x^{\gamma - i_{\ell_4} - \delta} + \sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{\gamma + k_{\ell_5} - \delta} + \epsilon,$$

for some $\epsilon \in K$. Finally, $z^\delta x^\gamma a y = \epsilon y \in P$.

Subcase b: Suppose $\sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} z^{i_{\ell_4}} \neq 0$. Then we can proceed similarly to Subcase a to get $y \in P$.

Subcase c: Suppose

$$\sum_{\ell_5=0}^{m_5} \alpha_{\ell_5} x^{k_{\ell_5}} = 0 = \sum_{\ell_4=0}^{m_4} \alpha_{\ell_4} z^{i_{\ell_4}}.$$

Subsubcase i: Suppose $\sum_{\ell_2=0}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y \neq 0$. Then $ay = \sum_{\ell_2=0}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y$. Let δ be the minimum exponent of z appearing in ay. Then

$$x^\delta ay = \sum_{\ell_2:\ell_2>\delta}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y + \epsilon y,$$

for some $\epsilon \in K$. Hence, $yx^\delta ay = \epsilon y \in P$.

Subsubcase ii: Suppose $\sum_{\ell_3=0}^{m_3} \alpha_{\ell_3} yx^{k_{\ell_3}} \neq 0$. Proceed similarly to Subsubcase i.

Subsubcase iii: Suppose

$$\sum_{\ell_2=0}^{m_2} \alpha_{\ell_2} z^{i_{\ell_2}} y = 0 = \sum_{\ell_3=0}^{m_3} \alpha_{\ell_3} yx^{k_{\ell_3}}.$$

Then let δ be the minimum exponent of z appearing in a. Then

$$x^\delta a = \sum_{\ell_1:i_{\ell_1}>\delta}^{m_1} \alpha_{\ell_1} z^{i_{\ell_1}-\delta} yx^{k_{\ell_1}} + \sum_{\ell_1:i_{\ell_1}=\delta}^{m_1} \alpha_{\ell_1} yx^{k_{\ell_1}}.$$

Hence,

$$yx^\delta a = \sum_{\ell_1:i_{\ell_1}=\delta}^{m_1} \alpha_{\ell_1} yx^{k_{\ell_1}}$$

and we can then proceed as in Subsubcase ii to get $y \in P$.

Therefore, any ideal of $R/\langle \lambda zx - \lambda - y\rangle$ contains y. □

Thus, the only nonzero prime ideals of R are those containing y and those of the form $\langle \lambda zx - \lambda + y\rangle$ for $\lambda \neq 0 \in K$. Thus, combining the above results yields:

(i) all primitive ideals and all nonzero prime ideals of R not containing y are one of the following: $\langle \lambda zx - \lambda + y\rangle$ for $\lambda \neq 0 \in K$,

(ii) a classification of the prime and primitive ideals of R containing y can be found in [1], and

(iii) as proven earlier, $\langle 0\rangle$ is a prime ideal of R.

Therefore, we may consider the classification of the prime ideals, primitive ideals, and irreducible representations of $K\{x_1, \ldots, x_n\}/\langle x_i x_j = \alpha_{ij}, 1 \leq i < j \leq n\rangle$ complete.

Appendix

Finite Dimensional Case

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Proposition .12. *Let $X_1, X_2, \cdots, X_m$, for $m \geq 2$, be linear operators on an n dimensional vector space over an algebraically closed field K and let $\alpha_{ij} \in K$ for all i and j. Let α_{ij} also denote the corresponding scalar operator. Suppose that $X_i X_j = \alpha_{ij}$ for all i, j where $1 \leq j < i \leq m$. Then $X_1, X_2, \cdots, X_m$ have a common eigenvector.*

Proof. **Case 1:** Suppose there exists an eigenvector v of X_1 with $X_1 v = \lambda_1 v$ and $\lambda_1 \neq 0$. Then for all $1 < \ell \leq m$,

$$X_\ell \cdot v = \lambda_1^{-1} X_\ell \cdot \lambda_1 v = \lambda_1^{-1} X_\ell X_1 \cdot v = \alpha_{\ell 1} \lambda_1^{-1} v.$$

Hence, for all $1 < \ell \leq m$, the vector v is an eigenvector of X_ℓ with eigenvalue $\alpha_{\ell 1} \lambda_1^{-1}$.

Case 2: Suppose all eigenvectors v of X_1 have eigenvalue 0. Then $X_\ell X_1 \cdot v = \alpha_{\ell 1} v$. But $X_\ell X_1 \cdot v = 0$, as well, since v is an eigenvector of X_1 with eigenvalue zero. Hence, $\alpha_{i1} = 0$ for all i.

Next, identify $X_1, \cdots, X_m$ with $n \times n$ matrices such that X_1 is in Jordan Canonical form. Since all eigenvalues of X_1 are zero, X_1 has zeros on the diagonal and zeros or ones on the super-diagonal. In fact, since $X_1 \neq 0$, there must be an i and a j such that the (i, j)-entry of X_1 is one. Let w be the vector in K^n with zeros everywhere, except a one in the jth-entry. Then $X_1 \cdot w \neq 0$. Note that X_1 is strictly upper-triangular and hence nilpotent. Therefore, there exists an h such that $X_1^h \cdot w \neq 0$, but $X_1^{h+1} \cdot w = 0$. This implies that $X_1^h \cdot w$ is an eigenvector with eigenvalue zero for X_1. Also, since $\alpha_{i1} = 0$ for all i, $X_1^h \cdot w$ is an eigenvector of X_ℓ for all ℓ (using the fact that $X_\ell X_1^h \cdot w = (X_\ell X_1)(X_1^{h-1} \cdot w)$). □

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