Research Statement

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1 Introduction

My research to date has primarily concerned the study of prime ideal theory and representation theory of finitely presented algebras of finite Gelfand-Kirillov dimension. Gelfand-Kirillov dimension, which has emerged as one of the important dimension functions, is the generalization of the Krull dimension of a commutative finitely generated algebra. Thus, we can think of Gelfand-Kirillov dimension as the dimension of the "noncommutative spaces" associated to noncommutative rings.

There have been numerous successful studies of prime ideals and representations of finitely generated noetherian algebras (e.g., quantum groups, enveloping algebras, and noetherian group algebras). Furthermore, in the 1940's, Jacobson determined the two-sided ideals and analyzed finitely generated modules of Ore extensions of the form $R[x; \phi]$ (for R a division ring and ϕ an endomorphism of R).

Irving, in the 1970's, studied the prime ideal structure of arbitrary Ore extensions $S = R[x; \phi, \delta]$ of a commutative noetherian ring R. Irving found that the prime ideals of S are closely related to the ϕ-prime ideals of R [3]. Since then, there have been many other successful studies of the representation theory of iterated Ore extensions (e.g., Goodearl and Letzter, Cauchon). In 2000, Gerritzen completely classified the irreducible representations of $R\{x, y\}/\langle yx - 1\rangle$ (which Jacobson originally studied in the 1950's) [1].

The algebras that I am studying in my dissertation are iterated Ore extensions, but provide new examples not covered by these previous studies. The algebras in my research are particularly interesting because, as filtered vector spaces, they are very "close" to commutative algebras. Thus these algebras can be viewed as natural generalizations of commutative polynomial algebras and as being related to quantum groups.

2 My Dissertation

2.1 Overview

In particular, my dissertation studies the prime spectra of algebras of the form

$$S = K\{x, y, z\}/\langle xy + \alpha_1 yx = \beta_1, xz + \alpha_2 zx = \beta_2, yz + \alpha_3 zy = \beta_3\rangle,$$

where K is an algebraically closed field and α_1, α_2, α_3, β_1, β_2, and β_3 are elements of K. Known work has shown that some algebras of this form have mathematically undesirable properties. For instance, $A = K\{x, y, z\}/\langle xy = 0, yz = 0, xz = 1\rangle$ has non-Goldie prime factors and infinite Krull dimension. However, the Gelfand-Kirillov dimension of A is 3.

I have essentially completed a classification of the prime spectrum of S. My work to date has shown the following:

Theorem 1: For any α_1, α_2, α_3, β_1, β_2, and β_3 in K, the prime spectrum of S has finite topological dimension under the Zariski topology.

Thus, while A has some undesirable properties, its prime spectrum is fairly well-behaved.

2.2 Methods

To classify the prime spectra of algebras of the form

$$K\{x, y, z\}/\langle xy + \alpha_1 yx = \beta_1, xz + \alpha_2 zx = \beta_2, yz + \alpha_3 zy = \beta_3\rangle,$$

I began by studying the 64 cases that occur when we allow the scalars to be zero or nonzero. Many of the cases are degenerate or the prime spectra have already been classified. My study reduced to the following cases (where the variables α_1, α_2, α_3, β_1, β_2, and β_3 now stand for nonzero elements of K):

1. $A = K\{x, y, z\}/\langle xy = 0, yz = 0, xz = 1\rangle$
2. $B = K\{x, y, z\}/\langle xy + \alpha_1 yx = \beta_1, xz + \alpha_2 zx = \beta_2, yz + \alpha_3 zy = \beta_3\rangle$
3. $C = K\{x, y, z\}/\langle xy + \alpha_1 yx = 0, xz + \alpha_2 zx = \beta_2, yz + \alpha_3 zy = 0\rangle$
4. $D = K\{x, y, z\}/\langle xy + \alpha_1 yx = 0, xz + \alpha_2 zx = \beta_2, yz + \alpha_3 zy = \beta_3\rangle$
5. $E = K\{x, y, z\}/\langle xy + \alpha_1 yx = \beta_1, xz + \alpha_2 zx = \beta_2, yz = 1\rangle$
6. $F = K\{x, y, z\}/\langle xy = \beta_1, xz + \alpha_2 zx = 0, yz + \alpha_3 zy = 0\rangle$

As a non-noetherian algebra, A is interesting to study on its own. I have completed a classification of the prime ideals and irreducible representations of A. I also generalized these results to include a complete classification of the prime ideals and irreducible representations of algebras of the form

$$T = K\{x_1, \ldots, x_n\}/\langle x_i x_j = \alpha_{ij}, 1 \leq i < j \leq n\rangle,$$

where K is an algebraically closed field and $\alpha_{ij} \in K$ for $1 \leq i < j \leq n$. This classification proves the following:

Theorem 2 For any choice of $\alpha_{ij} \in K$ for $1 \leq i < j \leq n$, the topological dimension of the prime spectrum of T under the Zariski topology is less than or equal to the Gelfand-Kirillov dimension of T.

Adam Berliner, as a participant in a Research Experience for Undergraduates under my thesis advisor Edward Letzter, proved a result that I used in the study of this case. As a corollary to Berliner's work, the finite dimensional representations of A are all one-dimensional.

I am currently preparing a paper on the classification of the prime ideals and irreducible representations of T for submission. I am also currently still researching whether or not the prime spectrum of A is noetherian, under the Zariski topology.

Finally, I have studied the remaining five cases. The classes B, C, and D are noetherian Ore extensions of a quantized Weyl algebra. I have essentially completed a classification of the prime spectra in these cases using [2]. My dissertation shows that E degenerates or is isomorphic to K depending upon relations between α_1 and α_2. I have finished a classification of the prime ideals of F. The work in this case is similar to the work on A. Since F is not noetherian, there is little existing theory. I am also still studying whether or not the prime spectrum of F is noetherian under the Zariski topology.

2.3 Future Work

In the future I would like to continue research in this area by studying the prime ideals and representation theory of some classes of algebras of the form $K\{x_1, \cdots, x_t\}/\langle$ relations of degree $\leq 2\rangle$ that are of finite Gelfand-Kirillov dimension. My thesis advisor, Edward Letzter, has had much success using topics related to my dissertation for undergraduate research (Research Experience for Undergraduates programs at Texas A&M University and Temple University). I would love the opportunity to lead undergraduates in research in this area as I continue similar work.

References

[1] L. Gerritzen. Modules over the algebra of the noncommutative equation $xy = 1$. *Archiv der Mathematik*, 75:98–112, 2000.

[2] K.R. Goodearl and E.S. Letzter. Prime ideals in skew and q-skew polynomial rings. *Memoirs of the American Mathematical Society*, 109:521, 1994.

[3] Ronald S. Irving. Prime ideals of ore extensions over commutative rings, ii. *Journal of Algebra*, 58:399–423, 1979.