# SNELL'S LAW
# AND UNIFORM REFRACTION

CRISTIAN E. GUTIÉRREZ

## Contents



## 1. Snell's law of refraction

1.1. **In vector form.** Suppose $\Gamma$ is a surface in $\mathbb{R}^3$ that separates two media I and II that are homogeneous and isotropic. Let $v_1$ and $v_2$ be the velocities of propagation of light in the media I and II respectively. The index of refraction of the medium I is by definition $n_1 = c/v_1$, where $c$ is the velocity of propagation of light in the vacuum, and similarly $n_2 = c/v_2$. If a ray of light* having direction $x \in S^2$ and

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*Since the refraction angle depends on the frequency of the radiation, we assume our light ray is monochromatic.

traveling through the medium I hits $\Gamma$ at the point $P$, then this ray is refracted in the direction $m \in S^2$ through the medium II according with the Snell law in vector form:

$$n_1(x \times \nu) = n_2(m \times \nu), \tag{1.1}$$

where $\nu$ is the unit normal to the surface to $\Gamma$ at $P$ going towards the medium II. This has several consequences:

(a) the vectors $x, m, \nu$ are all on the same plane (called plane of incidence);

(b) the well known Snell law in scalar form

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where $\theta_1$ is the angle between $x$ and $\nu$ (the angle of incidence), $\theta_2$ the angle between $m$ and $\nu$ (the angle of refraction),

From (1.1), $(n_1 x - n_2 m) \times \nu = 0$, which means that the vector $n_1 x - n_2 m$ is parallel to the normal vector $\nu$. If we set $\kappa = n_2/n_1$, then

$$x - \kappa\, m = \lambda \nu, \tag{1.2}$$

for some $\lambda \in \mathbb{R}$. Taking dot products we get $\lambda = \cos\theta_1 - \kappa\cos\theta_2$, $\cos\theta_1 = x\cdot\nu > 0$, and $\cos\theta_2 = m\cdot\nu = \sqrt{1-\kappa^{-2}[1-(x\cdot\nu)^2]}$, so

$$\lambda = x\cdot\nu - \kappa\,\sqrt{1-\kappa^{-2}(1-(x\cdot\nu)^2}. \tag{1.3}$$

Refraction behaves differently for $\kappa < 1$ and for $\kappa > 1$.

1.2. $\kappa < 1$**.** This means $v_1 < v_2$, and so waves propagate in medium II faster than in medium I, or equivalently, medium I is denser than medium II. In this case the refracted rays tend to bent away from the normal, that is the case for example, when medium I is glass and medium II is air. Indeed, from the scalar Snell law, $\sin\theta_1 = \kappa\,\sin\theta_2 < \sin\theta_2$ and so $\theta_1 < \theta_2$. For this reason, the maximum angle of refraction $\theta_2$ is $\pi/2$ which, from the Snell law in scalar form, is achieved when $\sin\theta_1 = n_2/n_1 = \kappa$. So there cannot be refraction when the incidence angle $\theta_1$ is beyond this critical value, that is, we must have $0 \leq \theta_1 \leq \theta_c = \arcsin \kappa$.[†] Once again from the Snell law in scalar form,

$$\theta_2 - \theta_1 = \arcsin(\kappa^{-1} \sin \theta_1) - \theta_1 \tag{1.4}$$

and it is easy to verify that this quantity is strictly increasing for $\theta_1 \in [0, \theta_c]$, and therefore $0 \leq \theta_2 - \theta_1 \leq \dfrac{\pi}{2} - \theta_c$. We then have $x \cdot m = \cos(\theta_2 - \theta_1) \geq \cos(\pi/2 - \theta_c) = \kappa$, and therefore obtain the following physical constraint for refraction:

(1.5) *if $\kappa = n_2/n_1 < 1$ and a ray of direction $x$ through medium I is refracted into medium II in the direction $m$, then $m \cdot x \geq \kappa$.*

Notice also that in this case $\lambda > 0$ in (1.3).

Conversely, given $x, m \in S^2$ with $x \cdot m \geq \kappa$ and $\kappa < 1$, it follows from (1.4) that there exists a hyperplane refracting any ray through medium I with direction $x$ into a ray of direction $m$ in medium II.

1.3. $\kappa > 1$**.** In this case, waves propagate in medium I faster than in medium II, and the refracted rays tend to bent towards the normal. By the Snell law, the maximum angle of refraction denoted by $\theta_c^*$ is achieved when $\theta_1 = \pi/2$, and $\theta_c^* = \arcsin(1/\kappa)$. Once again from the Snell law in scalar form

$$\theta_1 - \theta_2 = \arcsin(\kappa \sin \theta_2) - \theta_2 \tag{1.6}$$

which is strictly increasing for $\theta_2 \in [0, \theta_c^*]$, and $0 \leq \theta_1 - \theta_2 \leq \dfrac{\pi}{2} - \theta_c^*$. We therefore obtain the following physical constraint for the case $\kappa > 1$:

(1.7) *if a ray with direction $x$ traveling through medium I is refracted into a ray in medium II with direction $m$, then $m \cdot x \geq 1/\kappa$.*

Notice also that in this case $\lambda < 0$ in (1.3).

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[†] If $\theta_1 > \theta_c$, then the phenomenon of total internal reflection occurs.

On the other hand, by (1.6), if $x, m \in S^2$ with $x \cdot m \geq 1/\kappa$ and $\kappa > 1$, then there exists a hyperplane refracting any ray of direction $x$ through medium I into a ray with direction $m$ in medium II.

We summarize the above discussion on the physical constraints of refraction in the following lemma.

**Lemma 1.1.** *Let $n_1$ and $n_2$ be the indices of refraction of two media I and II, respectively, and $\kappa = n_2/n_1$. Then a light ray in medium I with direction $x \in S^2$ is refracted by some surface into a light ray with direction $m \in S^2$ in medium II if and only if $m \cdot x \geq \kappa$, when $\kappa < 1$; and if and only if $m \cdot x \geq 1/\kappa$, when $\kappa > 1$.*

1.4. $\kappa = 1$**.** This corresponds to reflection. It means

$$x - m = \lambda\, \nu. \tag{1.8}$$

Taking dot products with $x$ and then with $m$ yields $1 - m \cdot x = \lambda\, x \cdot \nu$ and $x \cdot m - 1 = \lambda\, m \cdot \nu$, then $x \cdot \nu = -m \cdot \nu$. Also taking dot product with $x$ in (1.8) then yields $\lambda = 2\, x \cdot \nu$. Therefore

$$m = x - 2(x \cdot \nu)\nu.$$

## 2. Uniform refraction

2.1. **Surfaces with the uniform refracting property: far field case.** Let $m \in S^2$ be fixed, and we ask the following: if rays of light emanate from the origin inside medium I, what is the surface $\Gamma$, interface of the media I and II, that refracts all these rays into rays parallel to $m$?

Suppose $\Gamma$ is parameterized by the polar representation $\rho(x)x$ where $\rho > 0$ and $x \in S^2$. Consider a curve on $\Gamma$ given by $r(t) = \rho(x(t))\, x(t)$ for $x(t) \in S^2$. According to (1.2), the tangent vector $r'(t)$ to $\Gamma$ satisfies $r'(t) \cdot (x(t) - \kappa\, m) = 0$. That is, $([\rho(x(t))]' x(t) + \rho(x(t)) x'(t)) \cdot (x(t) - \kappa\, m) = 0$, which yields $(\rho(x(t))(1 - \kappa m \cdot x(t)))' = 0$. Therefore

$$\rho(x) = \frac{b}{1 - \kappa\, m \cdot x} \tag{2.9}$$

for $x \in S^2$ and for some $b \in \mathbb{R}$. To understand the surface given by (2.9), we distinguish two cases $\kappa < 1$ and $\kappa > 1$.

2.2. $\kappa = 1$. When $\kappa = 1$ we see this is a paraboloid. Indeed, let $m = -e_n$, then a point $X = \rho(x)x$ is on the surface (2.9) if $|X| = b - x_n$. The distance from $X$ to the plane $x_n = b$ is $b - x_n$, and the distance from $X$ to 0 is $|X|$. So this is a paraboloid with focus at 0, directrix plane $x_n = b$ and axis in the direction $-e_n$.

2.3. $\kappa < 1$. For $b > 0$, we will see that the surface $\Gamma$ given by (2.9) is an ellipsoid of revolution about the axis of direction $m$. Suppose for simplicity that $m = e_n$, the $n$th-coordinate vector. If $y = (y', y_n) \in \mathbb{R}^n$ is a point on $\Gamma$, then $y = \rho(x)x$ with $x = y/|y|$. From (2.9), $|y| - \kappa\, y_n = b$, that is, $|y'|^2 + y_n^2 = (\kappa\, y_n + b)^2$ which yields $|y'|^2 + (1-\kappa^2)y_n^2 - 2\kappa b y_n = b^2$. This surface $\Gamma$ can be written in the form

$$\frac{|y'|^2}{\left(\dfrac{b}{\sqrt{1-\kappa^2}}\right)^2} + \frac{\left(y_n - \dfrac{\kappa b}{1-\kappa^2}\right)^2}{\left(\dfrac{b}{1-\kappa^2}\right)^2} = 1 \tag{2.10}$$

which is an ellipsoid of revolution about the $y_n$ axis with foci $(0, 0)$ and $(0, 2\kappa b/(1-\kappa^2))$. Since $|y| = \kappa y_n + b$ and the physical constraint for refraction (1.5), $\dfrac{y}{|y|} \cdot e_n \geq \kappa$ is equivalent to $y_n \geq \dfrac{\kappa b}{1-\kappa^2}$. That is, for refraction to occur $y$ must be in the upper part of the ellipsoid (2.10); we denote this semi-ellipsoid by $E(e_n, b)$. To verify that $E(e_n, b)$ has the uniform refracting property, that is, it refracts any ray emanating from the origin in the direction $e_n$, we check that (1.2) holds at each point. Indeed, if $y \in E(e_n, b)$, then $\left(\dfrac{y}{|y|} - \kappa e_n\right)\dfrac{y}{|y|} \geq 1-\kappa > 0$, and $\left(\dfrac{y}{|y|} - \kappa e_n\right)\cdot e_n \geq 0$, and so $\dfrac{y}{|y|} - \kappa e_n$ is an outward normal to $E(e_n, b)$ at $y$.

Rotating the coordinates, it is easy to see that the surface given by (2.9) with $\kappa < 1$ and $b > 0$ is an ellipsoid of revolution about the axis of direction $m$ with foci 0 and $\dfrac{2\kappa b}{1-\kappa^2} m$. Moreover, the semi-ellipsoid $E(m, b)$ given by

$$E(m, b) = \{\rho(x)x : \rho(x) = \frac{b}{1 - \kappa\, m \cdot x},\ x \in S^{n-1},\ x \cdot m \geq \kappa\}, \tag{2.11}$$

has the uniform refracting property, any ray emanating from the origin $O$ is refracted in the direction $m$.

2.4. $\kappa > 1$. Due to the physical constraint of refraction (1.7), we must have $b < 0$ in (2.9). Define for $b > 0$

$$H(m,b) = \{\rho(x)x : \rho(x) = \frac{b}{\kappa\, m \cdot x - 1},\ x \in S^{n-1},\ x \cdot m \geq 1/\kappa\}. \tag{2.12}$$

We claim that $H(m,b)$ is the sheet with opening in direction $m$ of a hyperboloid of revolution of two sheets about the axis of direction $m$. To prove the claim, set for simplicity $m = e_n$. If $y = (y', y_n) \in H(e_n, b)$, then $y = \rho(x)x$ with $x = y/|y|$. From (2.12), $\kappa\, y_n - |y| = b$, and therefore $|y'|^2 + y_n^2 = (\kappa\, y_n - b)^2$ which yields $|y'|^2 - (\kappa^2 - 1)\left[\left(y_n - \frac{\kappa b}{\kappa^2-1}\right)^2 - \left(\frac{\kappa b}{\kappa^2-1}\right)^2\right] = b^2$. Thus, any point $y$ on $H(e_n, b)$ satisfies the equation

$$\frac{\left(y_n - \frac{\kappa b}{\kappa^2-1}\right)^2}{\left(\frac{b}{\kappa^2-1}\right)^2} - \frac{|y'|^2}{\left(\frac{b}{\sqrt{\kappa^2-1}}\right)^2} = 1 \tag{2.13}$$

which represents a hyperboloid of revolution of two sheets about the $y_n$ axis with foci $(0,0)$ and $(0, 2\kappa b/(\kappa^2-1))$. Moreover, the upper sheet of this hyperboloid of revolution is given by

$$y_n = \frac{\kappa b}{\kappa^2-1} + \frac{b}{\kappa^2-1}\sqrt{1 + \frac{|y'|^2}{\left(b/\sqrt{\kappa^2-1}\right)^2}}$$

and satisfies $\kappa y_n - b > 0$, and hence has polar equation $\rho(x) = \dfrac{b}{\kappa\, e_n \cdot x - 1}$. Similarly, the lower sheet satisfies $\kappa y_n - b < 0$ and has polar equation $\rho(x) = \dfrac{b}{\kappa\, e_n \cdot x + 1}$. For a general $m$, by a rotation, we obtain that $H(m,b)$ is the sheet with opening in direction $m$ of a hyperboloid of revolution of two sheets about the axis of direction $m$ with foci $(0,0)$ and $\dfrac{2\kappa b}{\kappa^2-1} m$.

Notice that the focus $(0,0)$ is outside the region enclosed by $H(m,b)$ and the focus $\dfrac{2\kappa b}{\kappa^2-1}m$ is inside that region. The vector $\kappa m - \dfrac{y}{|y|}$ is an inward normal to $H(m,b)$ at $y$, because by (2.12)

$$\left(\kappa m - \frac{y}{|y|}\right)\cdot\left(\frac{2\kappa b}{\kappa^2-1}m - y\right) \geq \frac{2\kappa^2 b}{\kappa^2-1} - \frac{2\kappa b}{\kappa^2-1} - \kappa m\cdot y + |y|$$
$$= \frac{2\kappa b}{\kappa+1} - b = \frac{b(\kappa-1)}{\kappa+1} > 0.$$

Clearly, $\left(\kappa m - \dfrac{y}{|y|}\right)\cdot m \geq \kappa - 1$ and $\left(\kappa m - \dfrac{y}{|y|}\right)\cdot \dfrac{y}{|y|} > 0$. Therefore, $H(m,b)$ satisfies the uniform refraction property.

We remark that one has to use $H(-e_n,b)$ to uniformly refract in the direction $-e_n$, and due to the physical constraint (1.7), the lower sheet of the hyperboloid of equation (2.13) cannot refract in the direction $-e_n$.

From the above discussion, we have proved the following.

**Lemma 2.1.** *Let $n_1$ and $n_2$ be the indexes of refraction of two media I and II, respectively, and $\kappa = n_2/n_1$. Assume that the origin $O$ is inside medium I, and $E(m,b), H(m,b)$ are defined by* (2.11) *and* (2.12)*, respectively. We have:*

(i) *If $\kappa < 1$ and $E(m,b)$ is the interface of media I and II, then $E(m,b)$ refracts all rays emitted from $O$ into rays in medium II with direction $m$.*
(ii) *If $\kappa > 1$ and $H(m,b)$ separates media I and II, then $H(m,b)$ refracts all rays emitted from $O$ into rays in medium II with direction $m$.*

## 3. Uniform refraction: near field case

The question we ask is: given a point $O$ inside medium I and a point $P$ inside medium II, find an interface surface $\mathcal{S}$ between media I and II that refracts all rays emanating from the point $O$ into the point $P$. Suppose $O$ is the origin, and let $X(t)$ be a curve on $\mathcal{S}$. By the Snell law of refraction the tangent vector $X'(t)$ satisfies

$$X'(t)\cdot\left(\frac{X(t)}{|X(t)|} - \kappa\frac{P - X(t)}{|P - X(t)|}\right) = 0.$$

That is,

$$|X(t)|' + \kappa|P - X(t)|' = 0.$$

Therefore $\mathcal{S}$ is the Cartesian oval

$$|X| + \kappa|X - P| = b. \tag{3.14}$$

Since $f(X) = |X| + \kappa|X - P|$ is a convex function, the oval is a convex set.

Department of Mathematics, Temple University, Philadelphia, PA 19122

*E-mail address*: gutierre@temple.edu