OPTIMAL MASS TRANSPORT AND SOLUTION TO THE FAR FIELD REFRACTOR PROBLEM

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1. Optimal transportation set up

Let D, D^* be two domains on S^{n-1} or bounded domains in a manifold (D might be contained in one manifold and D^* in another) with $|\partial D| = 0$.

Let $\mathcal{N}$ be a multi-valued mapping from $\overline{D}$ onto $\overline{D^*}$ such that $\mathcal{N}(x)$ is single-valued a.e. on $\overline{D}$. For $F \subset \overline{D^*}$, we set

$$\mathcal{T}(F) = \mathcal{N}^{-1}(F) = \{x \in \overline{D} : \mathcal{N}(x) \cap F \neq \emptyset\}.$$

We say $\mathcal{N}$ is measurable if $\mathcal{T}(F)$ is Lebesgue measurable for any Borel set $F \subset \overline{D^*}$.

For example, if $\mathcal{N} = \partial u$ with u convex, then $\mathcal{N}$ is measurable (see exercises).

Given nonnegative $g \in L^1(D)$ and a finite Radon measure Γ on $\overline{D^*}$ satisfying $\int_{\overline{D}} g(x)\,dx = \Gamma(\overline{D^*}) > 0$, we say $\mathcal{N}$ is measure preserving from $g(x)dx$ to Γ if for any Borel $F \subset \overline{D^*}$

$$\int_{\mathcal{T}(F)} g(x)\,dx = \Gamma(F). \tag{1.1}$$

Lemma 1.1. *$\mathcal{N}$ is a measure preserving mapping from $g(x)dx$ to Γ if and only if for any $v \in C(\overline{D^*})$*

$$\int_{\overline{D}} v(\mathcal{N}(x))g(x)\,dx = \int_{\overline{D^*}} v(m)\,d\Gamma(m). \tag{1.2}$$

We remark that $v(\mathcal{N}(x))$ is well defined for $x \in \overline{D} \setminus S$ where $\mathcal{N}(x)$ is single-valued on $\overline{D} \setminus S$ and $|S| = 0$, and $\int_{\overline{D}} v(\mathcal{N}(x))g(x)\,dx$ is understood as $\int_{\overline{D}\setminus S} v(\mathcal{N}(x))g(x)\,dx$.

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Proof. Let $\mathcal{N}$ be a measure preserving mapping. To show (1.2), it suffices to prove it for $v = \chi_F$, the characteristic function of a Borel set F. It is easy to verify that $\chi_{\mathcal{T}(F)}(x) = \chi_F(\mathcal{N}(x))$ for $x \in \overline{D} \setminus S$. Therefore by (1.1)

$$\int_{\overline{D^*}} \chi_F(m)\, d\Gamma = \int_{\mathcal{T}(F)\cap(\overline{D}\setminus S)} g\, dx = \int_{\overline{D}\setminus S} \chi_F(\mathcal{N}(x)) g(x)\, dx.$$

To prove the converse, assume that (1.2) holds. We now show for any relatively open set G in $\overline{D^*}$

$$\int_{\mathcal{T}(G)} g\, dx \leq \Gamma(G). \tag{1.3}$$

Indeed, given a compact set $K \subset G$, choose $v \in C(\overline{D^*})$ such that $0 \leq v \leq 1$, $v = 1$ on K, and $v = 0$ outside G. By (1.2), one gets

$$\int_{\mathcal{T}(K)} g(x)\, dx \leq \int_{\overline{D}} v(\mathcal{N}(x)) g(x)\, dx \leq \Gamma(G),$$

and (1.3) follows from arbitrariness of K. Because a Borel set can be approximated by open sets, (1.3) is still valid for Borel sets F in $\overline{D^*}$. Noticing $(\mathcal{T}(F))^c \subset \mathcal{T}(F^c)$, to get the reverse inequality we apply (1.3) to $\overline{D^*} \setminus F$ and (1.1) follows. □

Consider the general cost function $c(x, m) \in Lip(\overline{D} \times \overline{D^*})$, the space of Lipschitz functions on $\overline{D} \times \overline{D^*}$, and *the set of admissible functions*

$$\mathcal{K} = \{(u, v) : u \in C(\overline{D}), v \in C(\overline{D^*}), u(x) + v(m) \leq c(x, m),\ \forall x \in D, \forall m \in D^*\}.$$

Define the *dual functional* I for $(u, v) \in C(\overline{D}) \times C(\overline{D^*})$

$$I(u, v) = \int_D u(x) g(x)\, dx + \int_{\overline{D^*}} v(m)\, d\Gamma,$$

and define the *c- and c^*- transforms*

$$u^c(m) = \inf_{x\in\overline{D}} [c(x, m) - u(x)], \quad m \in \overline{D^*}; \qquad v_c(x) = \inf_{m\in\overline{D^*}} [c(x, m) - v(m)], \quad x \in \overline{D}.$$

Definition 1.2. *A function $\phi \in C(\overline{D})$ is c-concave if for $x_0 \in \overline{D}$, there exist $m_0 \in \overline{D^*}$ and $b \in \mathbb{R}$ such that $\phi(x) \leq c(x, m_0) - b$ on $\overline{D}$ with equality held at $x = x_0$.*

Obviously v_c is c-concave for any $v \in C(\overline{D^*})$. We collect the following properties:

(1) For any $u \in C(\overline{D})$ and $v \in C(\overline{D^*})$, $v_c \in Lip(\overline{D})$ and $u^c \in Lip(\overline{D^*})$ with Lipschitz constants bounded uniformly by the Lipschitz constant of c. Indeed, (x_0 the point where the minimum is attained)

$$\begin{aligned} u^c(m_1) - u^c(m_2) &\le u^c(m_1) - (c(x_0, m_2) - u(x_0)) \\ &\le c(x_0, m_1) - u(x_0) - c(x_0, m_2) + u(x_0) \le K|m_1 - m_2|. \end{aligned}$$

(2) If $(u, v) \in \mathcal{K}$, then $v(m) \le u^c(m)$ and $u(x) \le v_c(x)$. Also $(v_c, v), (u, u^c) \in \mathcal{K}$.

(3) ϕ is c-concave iff $\phi = (\phi^c)_c$.

Indeed, if $\phi(x) \le c(x, m_0) - b$ on $\overline{D}$ and the equality holds at $x = x_0$, then $b = \phi^c(m_0)$. So $\phi(x_0) = c(x_0, m_0) - \phi^c(m_0)$ which yields $\phi(x_0) \ge (\phi^c)_c(x_0)$. On the other hand, from the definitions of c and c^* transforms we always have that $(\phi^c)_c \ge \phi$ for any ϕ.

Definition 1.3. *Given a function $\phi(x)$, the c-normal mapping of ϕ is defined by*

$$\mathcal{N}_{c,\phi}(x) = \{m \in \overline{D^*} : \phi(x) + \phi^c(m) = c(x, m)\}, \qquad \text{for } x \in \overline{D},$$

and $\mathcal{T}_{c,\phi}(m) = \mathcal{N}_{c,\phi}^{-1}(m) = \{x \in \overline{D} : m \in \mathcal{N}_{c,\phi}(x)\}$.

We assume that the cost function $c(x, m)$ satisfies the following:

$$\text{For any } c\text{-concave function } \phi,\ \mathcal{N}_{c,\phi}(x) \text{ is single-valued a.e. on } \overline{D} \text{ and } \mathcal{N}_{c,\phi} \text{ is Lebesgue measurable.} \tag{1.4}$$

Notice that if $c(x, m) = x \cdot m$, then $\mathcal{N}_{c,\phi}(x) = \partial^* \phi(x)$, where $\partial^* \phi$ is the super-differential of ϕ

$$\partial^* \phi(x) = \{m \in \mathbb{R}^n : \phi(y) \le \phi(x) + m \cdot (y - x)\ \forall\, y \in \Omega\},$$

and we have $\partial^* \phi(x) = -\partial(-\phi)(x)$.

Lemma 1.4. *Suppose that $c(x, m)$ satisfies the assumption* (1.4)*. Then*

(i) *If ϕ is c-concave and $\mathcal{N}_{c,\phi}$ is measure preserving from $g(x)dx$ to Γ, then (ϕ, ϕ^c) is a maximizer of $I(u, v)$ in $\mathcal{K}$.*

(ii) *If $\phi(x)$ is c-concave and (ϕ,ϕ^c) maximizes $I(u,v)$ in $\mathcal{K}$, then $\mathcal{N}_{c,\phi}$ is measure preserving from $g(x)dx$ to Γ.*

Proof. First prove (i). Given $(u,v)\in\mathcal{K}$, obviously

$$u(x)+v(\mathcal{N}_{c,\phi}(x))\leq c(x,\mathcal{N}_{c,\phi}(x))=\phi(x)+\phi^c(\mathcal{N}_{c,\phi}(x)),\qquad \text{a.e. } x \text{ on } \overline{D}.$$

Integrating the above inequality with respect to $g dx$ yields

$$\int_{\overline{D}} ug\,dx+\int_{\overline{D}} v(\mathcal{N}_{c,\phi}(x))g(x)\,dx\leq\int_{\overline{D}}\phi g\,dx+\int_{\overline{D}}\phi^c(\mathcal{N}_{c,\phi}(x))g(x)\,dx.$$

By Lemma 1.1, it yields $I(u,v)\leq I(\phi,\phi^c)$ and from (2) above the conclusion follows.

To prove (ii), let $\psi=\phi^c$, and for $v\in C(\overline{D^*})$, let $\psi_\theta(m)=\psi(m)+\theta\,v(m)$ where $0<|\theta|\leq\epsilon_0$ with ϵ_0 small, and let $\phi_\theta=(\psi_\theta)_c$. It suffices to show

$$0=\lim_{\theta\to0}\frac{I(\phi_\theta,\psi_\theta)-I(\phi,\psi)}{\theta}=\int_{\overline{D}}-v(\mathcal{N}_{c,\phi}(x))\,g\,dx+\int_{\overline{D^*}}v(m)\,d\Gamma. \tag{1.5}$$

Since $(\phi_\theta,\psi_\theta)\in\mathcal{K}$, $I(\phi_\theta,\psi_\theta)\leq I(\phi,\psi)$. So the limit must be zero if it exists. We have

$$\frac{I(\phi_\theta,\psi_\theta)-I(\phi,\psi)}{\theta}=\int_{\overline{D}}\frac{\phi_\theta-\phi}{\theta}\,g\,dx+\int_{\overline{D^*}}v(m)\,d\Gamma.$$

To prove (1.5), one only needs to show that $\dfrac{\phi_\theta(x)-\phi(x)}{\theta}$ is uniformly bounded and $\dfrac{\phi_\theta(x)-\phi(x)}{\theta}\to-v(\mathcal{N}_{c,\phi}(x))$ for all $x\in D\setminus S$ where $\mathcal{N}_{c,\phi}(x)$ is single-valued on $D\setminus S$ and $|S|=0$. Indeed, for $x\in D\setminus S$, we have $\phi_\theta(x)=c(x,m_\theta)-\psi_\theta(m_\theta)$ and $\phi(x)=c(x,m_1)-\psi(m_1)$ for some $m_\theta,m_1\in\overline{D^*}$. Then we get

$$-\theta\,v(m_\theta)\leq\phi_\theta(x)-\phi(x)\leq-\theta\,v(m_1).$$

Moreover, $m_1=\mathcal{N}_{c,\phi}(x)$ due to $\psi=\phi^c$. To finish the proof, we show that m_θ converges to m_1 as $\theta\to0$. Otherwise, there exists a sequence m_{θ_k} such that $m_{\theta_k}\to m_\infty\neq m_1$. So $\phi(x)=c(x,m_\infty)-\psi(m_\infty)$, which yields $m_\infty\in\mathcal{N}_{c,\phi}(x)$. We then get $m_1=m_\infty$, a contradiction. The proof is complete. □

Lemma 1.5. *There exists a c-concave ϕ such that*

$$I(\phi,\phi^c)=\sup\{I(u,v):(u,v)\in\mathcal{K}\}.$$

Proof. Let

$$I_0 = \sup\{I(u,v) : (u,v) \in \mathcal{K}\},$$

and let $(u_k, v_k) \in \mathcal{K}$ be a sequence such that $I(u_k, v_k) \to I_0$. Set $\bar{u}_k = (v_k)_c$ and $\bar{v}_k = (\bar{u}_k)^c$. From property (2) above, $(\bar{u}_k, \bar{v}_k) \in \mathcal{K}$ and $I(\bar{u}_k, \bar{v}_k) \to I_0$. Let $c_k = \min_{\overline{D}} \bar{u}_k$ and define

$$u_k^\sharp = \bar{u}_k - c_k, \qquad v_k^\sharp = \bar{v}_k + c_k.$$

Obviously $(u_k^\sharp, v_k^\sharp) \in \mathcal{K}$ and by the mass conservation of $g dx$ and Γ, $I(\bar{u}_k, \bar{v}_k) = I(u_k^\sharp, v_k^\sharp)$. Since $\bar{u}_k$ are uniformly Lipschitz, $u_k^\sharp$ are uniformly bounded. In addition, $v_k^\sharp = (\bar{u}_k)^c + c_k = (u_k^\sharp)^c$ and consequently $v_k^\sharp$ are also uniformly bounded. By Arzelá-Ascoli's theorem, $(u_k^\sharp, v_k^\sharp)$ contains a subsequence converging uniformly to (ϕ, ψ) on $\overline{D} \times \overline{D^*}$. We then obtain that $(\phi, \psi) \in \mathcal{K}$ and $I_0 = \sup\{I(u,v) : (u,v) \in \mathcal{K}\} = I(\phi, \psi)$. Notice that this shows in particular that the supremum of I over $\mathcal{K}$ is finite. From property (2) above, $(\psi_c, (\psi_c)^c)$ is the sought maximizer of $I(u,v)$, and ψ_c is c-concave. □

Lemma 1.6. *Suppose that $c(x,m)$ satisfies the assumption* (1.4). *Let (ϕ, ϕ^c) with ϕ c-concave be a maximizer of $I(u,v)$ in $\mathcal{K}$. Then $\inf_{s \in \mathcal{S}} \int_{\overline{D}} c(x, s(x)) g(x)\, dx$ is attained at $s = \mathcal{N}_{c,\phi}$, where $\mathcal{S}$ is the class of measure preserving mappings from $g(x)dx$ to Γ. Moreover*

$$\inf_{s \in \mathcal{S}} \int_{\overline{D}} c(x, s(x)) g(x)\, dx = \sup\{I(u,v) : (u,v) \in \mathcal{K}\}. \tag{1.6}$$

Proof. Let $\psi = \phi^c$. For $s \in \mathcal{S}$, we have

$$\begin{aligned}
\int_{\overline{D}} c(x, s(x)) g(x)\, dx &\geq \int_{\overline{D}} \Big(\phi(x) + \psi(s(x))\Big) g(x)\, dx \\
&= \int_{\overline{D}} \phi(x) g(x)\, dx + \int_{\overline{D}} \psi(s(x)) g(x)\, dx \\
&= \int_{\overline{D}} \phi(x) g(x)\, dx + \int_{\overline{D^*}} \psi(m)\, d\Gamma = I(\phi, \psi) \\
&= \int_{\overline{D}} \Big(\phi(x) + \psi(\mathcal{N}_{c,\phi}(x))\Big) g(x)\, dx, \text{ from Lemma 1.4(ii)} \\
&= \int_{\overline{D}} c(x, \mathcal{N}_{c,\phi}(x)) g(x)\, dx.
\end{aligned}$$

□

Obviously, for any c-concave function ϕ, $\mathcal{N}_{c,\phi}$ has the following converging property (C): if $m_k \in \mathcal{N}_{c,\phi}(x_k)$, $x_k \longrightarrow x_0$ and $m_k \longrightarrow m_0$, then $m_0 \in \mathcal{N}_{c,\phi}(x_0)$.

Lemma 1.7. *Assume that $c(x,m)$ satisfies the assumption* (1.4) *and that $\int_G g\,dx > 0$ for any open $G \subset D$. Then the minimizing mapping of $\inf_{s\in\mathcal{S}} \int_{\overline{D}} c(x,s(x))g(x)\,dx$ is unique in the class of measure preserving mappings from $g(x)dx$ to Γ with the converging property (C).*

Proof. From Lemmas 1.5 and 1.6, let $\mathcal{N}_{c,\phi}$ be a minimizing mapping associated with a maximizer (ϕ,ϕ^c) of $I(u,v)$ with ϕ c-concave. Suppose that $\mathcal{N}_0$ is another minimizing mapping with the converging property (C). Clearly

$$\int_{\overline{D}} \Big(c(x,\mathcal{N}_0(x)) - \phi(x) - \phi^c(\mathcal{N}_0(x))\Big)\,g(x)\,dx$$
$$= \inf_{s\in\mathcal{S}} \int_{\overline{D}} c(x,s(x))g(x)\,dx - \left(\int_{\overline{D}} \phi(x)g(x)\,dx + \int_{\overline{D^*}} \phi^c(m)\,d\Gamma\right) = 0,$$

and since $\phi(x)+\phi^c(\mathcal{N}_0(x)) \le c(x,\mathcal{N}_0(x))$, it follows that $\phi(x)+\phi^c(\mathcal{N}_0(x)) = c(x,\mathcal{N}_0(x))$ on the set $\{x \in D : g(x) > 0\}$ which is dense in D. Hence from (1.4) and the converging property (C), we get $\mathcal{N}_0(x) = \mathcal{N}_{c,\phi}(x)$ a.e. on D. □

We remark from the above proof that if $g(x) > 0$ on D, then the minimizing mapping of $\inf_{s\in\mathcal{S}} \int_{\overline{D}} c(x,s(x))g(x)\,dx$ is unique in the class of measure preserving mappings from $g(x)dx$ to Γ.

2. The refractor problem $\kappa < 1$

Let n_1 and n_2 be the indexes of refraction of two homogeneous and isotropic media I and II, respectively. Suppose that from a point O inside medium I light emanates with intensity $f(x)$ for $x \in \Omega$. We want to construct a refracting surface $\mathcal{R}$ parameterized as $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, separating media I and II, and such that all rays refracted by $\mathcal{R}$ into medium II have directions in Ω^* and the prescribed illumination intensity received in the direction $m \in \Omega^*$ is $f^*(m)$.

We first introduce the notions of refractor mapping and measure, and weak solution. In the next section we then convert the refractor problem into an optimal mass transport problem from $\overline{\Omega}$ to $\overline{\Omega^*}$ with the cost function $\log \frac{1}{1-\kappa\, x\cdot m}$ and establish existence and uniqueness of weak solutions.

Let Ω, Ω^* be two domains on S^{n-1}, the illumination intensity of the emitting beam is given by nonnegative $f(x) \in L^1(\overline{\Omega})$, and the prescribed illumination intensity of the refracted beam is given by a nonnegative Radon measure μ on $\overline{\Omega^*}$. Throughout this section, we assume that $|\partial\Omega| = 0$ and the physical constraint

$$\inf_{x\in\Omega, m\in\Omega^*} x\cdot m \geq \kappa. \tag{2.1}$$

We further suppose that the total energy conservation

$$\int_\Omega f(x)\,dx = \mu(\overline{\Omega^*}) > 0, \tag{2.2}$$

and for any open set $G \subset \Omega$

$$\int_G f(x)\,dx > 0, \tag{2.3}$$

where dx denotes the surface measure on S^{n-1}.

2.1. **Refractor measure and weak solutions.** We begin with the notions of refractor and supporting semi-ellipsoid.

Definition 2.1. *A surface $\mathcal{R}$ parameterized by $\rho(x)x$ with $\rho \in C(\overline{\Omega})$ is a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$ for the case $\kappa < 1$ (often simply called as refractor in this section) if for any $x_0 \in \overline{\Omega}$ there exists a semi-ellipsoid $E(m, b)$ with $m \in \overline{\Omega^*}$ such that $\rho(x_0) = \frac{b}{1-\kappa\, m\cdot x_0}$ and $\rho(x) \leq \frac{b}{1-\kappa\, m\cdot x}$ for all $x \in \overline{\Omega}$. Such $E(m, b)$ is called a supporting semi-ellipsoid of $\mathcal{R}$ at the point $\rho(x_0)x_0$.*

From the definition, any refractor is globally Lipschitz on $\overline{\Omega}$.

Definition 2.2. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the refractor mapping of $\mathcal{R}$ is the multi-valued map defined by for $x_0 \in \overline{\Omega}$*

$$\mathcal{N}_{\mathcal{R}}(x_0) = \{m \in \overline{\Omega^*} :\ E(m, b) \text{ supports } \mathcal{R} \text{ at } \rho(x_0)x_0 \text{ for some } b > 0\}.$$

Given $m_0 \in \overline{\Omega^*}$, the tracing mapping of $\mathcal{R}$ is defined by

$$\mathcal{T}_{\mathcal{R}}(m_0) = \mathcal{N}_{\mathcal{R}}^{-1}(m_0) = \{x \in \overline{\Omega} : \; m_0 \in \mathcal{N}_{\mathcal{R}}(x)\}.$$

Definition 2.3. *Given a refractor $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$, the Legendre transform of $\mathcal{R}$ is defined by*

$$\mathcal{R}^* = \{\rho^*(m)m : \; \rho^*(m) = \inf_{x\in\overline{\Omega}} \frac{1}{\rho(x)(1-\kappa\, x\cdot m)}, \; m \in \overline{\Omega^*}\}.$$

We now give some basic properties of Legendre transforms.

Lemma 2.4. *Let $\mathcal{R}$ be a refractor from $\overline{\Omega}$ to $\overline{\Omega^*}$. Then*

(i) *$\mathcal{R}^*$ is a refractor from $\overline{\Omega^*}$ to $\overline{\Omega}$.*
(ii) *$\mathcal{R}^{**} = (\mathcal{R}^*)^* = \mathcal{R}$.*
(iii) *If $x_0 \in \overline{\Omega}$ and $m_0 \in \overline{\Omega^*}$, then $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$ iff $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$.*

Proof. Given $m_0 \in \overline{\Omega^*}$, $\rho(x)(1-\kappa x\cdot m_0)$ must attain the maximum over $\overline{\Omega}$ at some $x_0 \in \overline{\Omega}$. Then $\rho^*(m_0) = 1/[\rho(x_0)(1-\kappa x_0\cdot m_0)]$. We always have

$$\rho^*(m) = \inf_{x\in\overline{\Omega}} \frac{1}{\rho(x)(1-\kappa m\cdot x)} \leq \frac{1}{\rho(x_0)(1-\kappa x_0\cdot m)}, \qquad \forall m \in \overline{\Omega^*}. \tag{2.4}$$

Hence $E(x_0, 1/\rho(x_0))$ is a supporting semi-ellipsoid to $\mathcal{R}^*$ at $\rho^*(m_0)m_0$. Thus, (i) is proved.

To prove (ii), from the definitions of Legendre transform and refractor mapping we have

$$\rho(x_0)\,\rho^*(m_0) = \frac{1}{1-\kappa m_0\cdot x_0} \quad \text{for } m_0 \in \mathcal{N}_{\mathcal{R}}(x_0). \tag{2.5}$$

For $x_0 \in \overline{\Omega}$, there exists $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$ and so from (2.5) $\rho^*(m_0) = \dfrac{1/\rho(x_0)}{1-\kappa x_0\cdot m_0}$. By (2.4), $\rho^*(m)(1-kx_0\cdot m)$ attains the maximum $1/\rho(x_0)$ at m_0. Thus,

$$\rho^{**}(x_0) = \inf_{m\in\overline{\Omega^*}} \frac{1}{\rho^*(m)(1-kx_0\cdot m)} = \frac{1}{\rho(x_0)^{-1}}.$$

To prove (iii), we get from the proof of (ii) that if $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$, then the semi-ellipsoid $E(x_0, 1/\rho(x_0))$ supports $\mathcal{R}^*$ at $\rho^*(m_0)m_0$ and so $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$. On the other hand, if $x_0 \in \mathcal{N}_{\mathcal{R}^*}(m_0)$, we get that $m_0 \in \mathcal{N}_{\mathcal{R}^{**}}(x_0)$, and since $\mathcal{R}^{**} = \mathcal{R}$, $m_0 \in \mathcal{N}_{\mathcal{R}}(x_0)$. □

The next two lemmas discuss the refractor measure.

Lemma 2.5. *$\mathcal{C} = \{F \subset \overline{\Omega^*} : \mathcal{T}_{\mathcal{R}}(F)$ is Lebesgue measurable$\}$ is a σ-algebra containing all Borel sets in $\overline{\Omega^*}$.*

Proof. Obviously, $\mathcal{T}_{\mathcal{R}}(\emptyset) = \emptyset$ and $\mathcal{T}_{\mathcal{R}}(\overline{\Omega^*}) = \overline{\Omega}$. Since $\mathcal{T}_{\mathcal{R}}(\cup_{i=1}^{\infty} F_i) = \cup_{i=1}^{\infty} \mathcal{T}_{\mathcal{R}}(F_i)$, $\mathcal{C}$ is closed under countable unions. Clearly for $F \subset \overline{\Omega^*}$

$$\begin{aligned} \mathcal{T}_{\mathcal{R}}(F^c) &= \{x \in \overline{\Omega} : \mathcal{N}_{\mathcal{R}}(x) \cap F^c \neq \emptyset\} \\ &= \{x \in \overline{\Omega} : \mathcal{N}_{\mathcal{R}}(x) \cap F = \emptyset\} \cup \{x \in \overline{\Omega} : \mathcal{N}_{\mathcal{R}}(x) \cap F^c \neq \emptyset,\ \mathcal{N}_{\mathcal{R}}(x) \cap F \neq \emptyset\} \\ &= [\mathcal{T}_{\mathcal{R}}(F)]^c \cup [\mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F)]. \end{aligned} \tag{2.6}$$

If $x \in \mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F) \cap \Omega$, then $\mathcal{R}$ parameterized by ρ has two distinct supporting semi-ellipsoids $E(m_1, b_1)$ and $E(m_2, b_2)$ at $\rho(x)x$. We show that $\rho(x)x$ is a singular point of $\mathcal{R}$. Otherwise, if $\mathcal{R}$ has the tangent hyperplane Π at $\rho(x)x$, then Π must coincide both with the tangent hyperplane of $E(m_1, b_1)$ and that of $E(m_2, b_2)$ at $\rho(x)x$. It follows from the Snell law that $m_1 = m_2$. Therefore, the area measure of $\mathcal{T}_{\mathcal{R}}(F^c) \cap \mathcal{T}_{\mathcal{R}}(F)$ is 0. So $\mathcal{C}$ is closed under complements, and we have proved that $\mathcal{C}$ is a σ-algebra.

To prove that $\mathcal{C}$ contains all Borel subsets, it suffices to show that $\mathcal{T}_{\mathcal{R}}(K)$ is compact if $K \subset \overline{\Omega^*}$ is compact. Let $x_i \in \mathcal{T}_{\mathcal{R}}(K)$ for $i \geq 1$. There exists $m_i \in \mathcal{N}_{\mathcal{R}}(x_i) \cap K$. Let $E(m_i, b_i)$ be the supporting semi-ellipsoid to $\mathcal{R}$ at $\rho(x_i)x_i$. We have

$$\rho(x)(1 - \kappa m_i \cdot x) \leq b_i \qquad \text{for } x \in \overline{\Omega}, \tag{2.7}$$

where the equality in (2.7) occurs at $x = x_i$. Assume that $a_1 \leq \rho(x) \leq a_2$ on $\overline{\Omega}$ for some constants $a_2 \geq a_1 > 0$. By (2.7) and (2.1), $a_1(1-\kappa) \leq b_i \leq a_2(1-\kappa^2)$. Assume through subsequence that $x_i \longrightarrow x_0$, $m_i \longrightarrow m_0 \in K$, $b_i \longrightarrow b_0$, as $i \longrightarrow \infty$. By taking limit in (2.7), one obtains that the semi-ellipsoid $E(m_0, b_0)$ supports $\mathcal{R}$ at $\rho(x_0)x_0$ and $x_0 \in \mathcal{T}_{\mathcal{R}}(m_0)$. This proves $\mathcal{T}_{\mathcal{R}}(K)$ is compact. □

Lemma 2.6. *Given a nonnegative $f \in L^1(\overline{\Omega})$, the set function*

$$\mathcal{M}_{\mathcal{R},f}(F) = \int_{\mathcal{T}_{\mathcal{R}}(F)} f\,dx$$

is a finite Borel measure defined on C and is called the refractor measure associated with $\mathcal{R}$ and f.

Proof. Let $\{F_i\}_{i=1}^\infty$ be a sequence of pairwise disjoint sets in C. Let $H_1 = \mathcal{T}_\mathcal{R}(F_1)$, and $H_k = \mathcal{T}_\mathcal{R}(F_k) \setminus \cup_{i=1}^{k-1}\mathcal{T}_\mathcal{R}(F_i)$, for $k \geq 2$. Since $H_i \cap H_j = \emptyset$ for $i \neq j$ and $\cup_{k=1}^\infty H_k = \cup_{k=1}^\infty \mathcal{T}_\mathcal{R}(F_k)$, it is easy to get

$$\mathcal{M}_{\mathcal{R},f}(\cup_{k=1}^\infty F_k) = \int_{\cup_{k=1}^\infty H_k} f\,dx = \sum_{k=1}^\infty \int_{H_k} f\,dx.$$

Observe that $\mathcal{T}_\mathcal{R}(F_k) \setminus H_k = \mathcal{T}_\mathcal{R}(F_k) \cap (\cup_{i=1}^{k-1}\mathcal{T}_\mathcal{R}(F_i))$ is a subset of the singular set of $\mathcal{R}$ and has area measure 0 for $k \geq 2$. Therefore, $\int_{H_k} f\,dx = \mathcal{M}_{\mathcal{R},f}(F_k)$ and the σ-additivity of $\mathcal{M}_{\mathcal{R},f}$ follows. □

The notion of weak solutions is introduced through the conservation of energy.

Definition 2.7. *A refractor $\mathcal{R}$ is a weak solution of the refractor problem for the case $\kappa < 1$ with emitting illumination intensity $f(x)$ on $\overline{\Omega}$ and prescribed refracted illumination intensity μ on $\overline{\Omega^*}$ if for any Borel set $F \subset \overline{\Omega^*}$*

$$\mathcal{M}_{\mathcal{R},f}(F) = \int_{\mathcal{T}_\mathcal{R}(F)} f\,dx = \mu(F). \tag{2.8}$$

2.2. **Solution of the refractor problem.** We introduce the cost

$$c(x,m) = \frac{1}{1 - \kappa\, x \cdot m}$$

for $x \in \Omega$ and $m \in \Omega^*$ where we assume $\Omega \cdot \Omega^* \geq \kappa$. From Definitions 1.2 and 2.1, $\mathcal{R} = \{\rho(x)x : x \in \overline{\Omega}\}$ is a refractor iff $\log\rho$ is c-concave. Using Definitions 1.3 and 2.2 we get that

$$\mathcal{N}_{c,\phi}(x) = \mathcal{N}_\mathcal{R}(x), \qquad \mathcal{R} = \{\rho(x)x : x \in \Omega\}, \qquad \rho(z) = e^{\phi(z)}.$$

Furthermore, $\log\rho^* = (\log\rho)^c$, $\log\rho = (\log\rho^*)_c$ by Remark (3) after Definition 1.2, and $\mathcal{N}_\mathcal{R}(x_0) = \mathcal{N}_{c,\log\rho}(x_0)$ by (2.5). By the Snell law and Lemma 2.5, $c(x,m)$ satisfies (1.4). From the definitions, $\mathcal{R}$ is a weak solution of the refractor problem iff $\log\rho$ is c-concave and $\mathcal{N}_{c,\log\rho}$ is a measure preserving mapping from $f(x)dx$ to μ.

By Lemma 1.5, there exists a c-concave $\phi(x)$ such that (ϕ, ϕ^c) maximizes

$$I(u,v) = \int_{\overline{\Omega}} uf\,dx + \int_{\overline{\Omega^*}} v\,d\mu(m)$$

in $\mathcal{K} = \{(u,v) \in C(\overline{\Omega}) \times C(\overline{\Omega^*}) : u(x) + v(m) \le c(x,m),\ \text{for } x \in \overline{\Omega}, m \in \overline{\Omega^*}\}$. Then by Lemma 1.4, $\mathcal{N}_{c,\phi}(x)$ is a measure preserving mapping from $f dx$ to μ. Therefore, $\mathcal{R} = \{e^{\phi(x)}x : x \in \overline{\Omega}\}$ is a weak solution of the refractor problem.

It remains to prove the uniqueness of solutions up to dilations. Let $\mathcal{R}_i = \{\rho_i(x)x : x \in \overline{\Omega}\}$, $i = 1, 2$, be two weak solutions of the refractor problem. Obviously, $\mathcal{N}_{c,\log\rho_i}$ have the converging property (C) stated before Lemma 1.7. It follows from Lemmas 1.4, 1.6 and 1.7 that $\mathcal{N}_{c,\log\rho_1}(x) = \mathcal{N}_{c,\log\rho_2}(x)$ a.e. on Ω. That is, $\mathcal{N}_{\mathcal{R}_1}(x) = \mathcal{N}_{\mathcal{R}_2}(x)$ a.e. on Ω. From the Snell law $\nu_i(x) = \dfrac{x - \kappa\mathcal{N}_{\mathcal{R}_i}(x)}{|x - \kappa\mathcal{N}_{\mathcal{R}_i}(x)|}$ is the unit normal to $\mathcal{R}_i$ towards medium II at $\rho_i(x)x$ where $\mathcal{R}_i$ is differentiable. So $\nu_1(x) = \nu_2(x)$ a.e. and consequently $\rho_1(x) = C\,\rho_2(x)$ for some $C > 0$.

References

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