Monge Ampère equations
Prof. Gutiérrez
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- $\Omega \subset \mathbb{R}^n$ is an open domain; $u : \Omega \to \mathbb{R}$; the subdifferential of u in Ω is the multivalued map
$$\partial u(y) = \{p \in \mathbb{R}^n : u(x) \geq u(y) + p \cdot (x - y) \,\forall x \in \Omega\}.$$
- The Legendre transformation of u is $u^*(p) = \sup_{x\in\Omega} (x \cdot p - u(x))$.
- Aleksandrov's lemma states that the set
$$\{p \in \mathbb{R}^n : \text{there exist } x, y \in \Omega \text{ such that } x \neq y \text{ and } p \in \partial u(x) \cap \partial u(y)\}$$
has Lebesgue measure zero.

1. Let $F(A) = \det A$ for each $n \times n$ symmetric matrix A. Prove that $\dfrac{\partial F}{\partial a_{ij}}(A) = A^{ij}$ where $\{A^{ij}\}$ is the cofactor matrix of A.

2. Let A be symmetric and non negative definite. Prove that
$$(\det A)^{1/n} = \inf\left\{\frac{1}{n}\text{trace}(AB) : B \text{ is symmetric and } \det B = 1\right\}.$$
HINT: if A, B are symmetric and positive definite, then $AB = O^tDO$ with O orthogonal and D diagonal, then $\text{trace}(AB) = \text{trace}D$; and next use the arithmetic-geometric inequality. To show equality assume A is diagonal and use Lagrange multipliers.

3. Deduce from Problem 2 the following inequality due to Minkowski: if A, B are symmetric non negative definite $n \times n$ matrices, then
$$(\det(A + B))^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n};$$
that is, the function $\det A$ is concave over the non negative definite symmetric matrices. Equality happens iff A is a multiple of B.

   HINT: for the equality assume $\det A > 0$. Multiply the identity by $(\det A^{-1})^{1/n}$ to obtain $\det(Id + BA^{-1})^{1/n} = 1 + \det(BA^{-1})^{1/n}$, next diagonalize BA^{-1} and show that all its eigenvalues are the same.

4. Let A be an $n \times n$ matrix and $u \in C^2$. Let $v(x) = u(Ax)$. Prove that $D^2v(x) = A^t\left((D^2u)(Ax)\right)A$. Therefore $\det D^2v(x) = (\det A)^2 \det(D^2u)(Ax)$.

5. Let $R, h > 0$ and $u(x) = h\dfrac{|x - x_0|}{R}$, that is, the graph of u is an upside down right cone with vertex at x_0 of height h and top $B_R(x_0) \times \{h\}$. Prove that
$$\partial u(x) = \begin{cases} \overline{B_{h/R}(0)}, & \text{if } x = x_0 \\ \dfrac{h}{R}\dfrac{x - x_0}{|x - x_0|}, & \text{if } x \in B_R(x_0) \setminus \{x_0\}. \end{cases}$$

6. Prove that $\partial u(x_0)$ is a convex set.

7. Suppose Ω is a bounded convex domain and u is bounded in Ω, so u^* is finite in $\mathbb{R}^n$. Prove that

   (a) $p_0 \in \partial u(x_0)$ iff $u(x_0) + u^*(p_0) = x_0 \cdot p_0$;

   (b) if $p_0 \in \partial u(x_0)$, then $x_0 \in \partial u^*(p_0)$;

   (c) we have $(u^*)^*(x) \leq u(x)$ for all $x \in \Omega$;

   (d) if $\partial u(x_0) \neq \emptyset$, then $(u^*)^*(x_0) = u(x_0)$;

   (e) if u is convex in Ω, then $p_0 \in \partial u(x_0)$ iff $x_0 \in \partial u^*(p_0)$;

   (f) $\partial u(x)$ is a singleton for a.e. $x \in \Omega$ (use Aleksandrov).

   (h) if $u : \Omega \to \mathbb{R}$ is convex, then for each Borel set $F \subset \mathbb{R}^n$ the set

   $$(\partial u)^{-1}(F) = \{x \in \Omega : \partial u(x) \cap F \neq \emptyset\}$$

   is Lebesgue measurable. HINT: use (e) and that from (h) ∂u is one-to-one except on a set of measure zero.

   (i) if $u : \Omega \to \mathbb{R}$ is convex and differentiable in Ω, then

   $$u^*(Du(x)) = x \cdot Du(x) - u(x)$$

   for each $x \in \Omega$.

   (j) if $\mathcal{N}u(x) = \{p \in \mathbb{R}^n : u(x) + u^*(p) = x \cdot p\}$, then $\mathcal{N}u(x) = \partial u(x)$.

8. Prove that the subdifferential is a monotone function, that is, for all $x_1, x_2 \in \Omega$ and $p_i \in \partial u(x_i)$, $i = 1, 2$, we have $(x_1 - x_2) \cdot (p_1 - p_2) \geq 0$.

9. Let T be a multivalued map defined for each $x \in \Omega$ such that $Tx \subset X$ where X is a fixed set. Prove that for each $E \subset \Omega$ we have

$$T(\Omega \setminus E) = (T(\Omega) \setminus T(E)) \cup (T(\Omega \setminus E) \cap T(E)).$$

10. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an inversible affine transformation, i.e., $Tx = Ax + b$ with A an $n \times n$ matrix, with $\det A \neq 0$, and $b \in \mathbb{R}^n$. Let $u : \Omega \to \mathbb{R}$, and set $v(x) = u(Tx)$ for $x \in T^{-1}\Omega$. Prove that

$$\partial v(x) = A^t\,(\partial u(Tx))$$

   for all $x \in T^{-1}\Omega$.

11. Let E be an ellipsoid in $\mathbb{R}^n$ with center at the origin, and consider in $\mathbb{R}^{n+1}$ the upside down right cone with vertex at the origin with height h and top $E \times \{h\}$. Let v be the function from E to $\mathbb{R}$ whose graph is this cone. Use problems 5 and 10 to calculate the subdifferential of v.

12. Let $B = B_R(0)$ be a ball in $\mathbb{R}^n$ and let p_0 be a point outside B. Let C be the convex hull of p_0 and B (an ice cream cone). Prove that the volume of C equals $c_n\, R^{n-1}\, |p_0|$.

13. Let $f \in C(\Omega)$ with Ω convex, and $R : \mathbb{R}^n \to \mathbb{R}$ continuous and both functions being positive. Suppose that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ convex solves the problem $\det D^2u(x) = \dfrac{f(x)}{R(Du(x))}$. If $R(p) = (1+|p|^2)^{(n+2)/2}$, then we get the Gauss curvature equation.

    Then prove that for each Borel set $E \subset \Omega$ we have

    $$\int_E f(x)\, dx = \int_{\partial u(E)} R(p)\, dp,$$

    and in particular, $\int_\Omega f(x)\, dx \le \int_{\mathbb{R}^n} R(p)\, dp$.

    HINT: use Sard's lemma: if $\Omega \subset \mathbb{R}^n$ is open, $g : \Omega \to \mathbb{R}^n$ is $C^1(\Omega)$, and $S_0 = \{x \in \Omega : \det g'(x) = 0\}$, then $g(S_0)$ has Lebesgue measure zero. Consider the set $A = \{x \in \Omega : D^2u(x) > 0\}$ and prove that Du is one to one on A. For that use that from the convexity of u we have that if $x_i \in A$, $i = 1, 2$, then $u(x_i) \ge u(x_j) + Du(x_j) \cdot (x_i - x_j)$ for $i \ne j$. If $Du(x_1) = Du(x_2)$, next use the Taylor integral formula of second order and conclude that $x_1 = x_2$. Apply Sard's lemma with $g = Du$ and use the formula of change of variables.

14. Let u, v smooth convex functions in Ω both vanishing on $\partial\Omega$. Suppose that $|1 - \det D^2u(x)| \le \epsilon$ for all $x \in \Omega$, and $\det D^2 v = 1$ in Ω. Prove that $|u(x) - v(x)| \le c_n\, \epsilon$ for all $x \in \Omega$.

    HINT: comparison principle.

15. An ellipsoid in $\mathbb{R}^n$ with center at x_0 is a set of the form

    $$E = \{x \in \mathbb{R}^n : \langle A(x - x_0), x - x_0\rangle \le 1\},$$

    where A is an $n \times n$ positive definite symmetric matrix and $\langle\, ,\rangle$ denotes the Euclidean inner product. Prove that

    $$|E| = \frac{\omega_n}{(\det A)^{1/2}},$$

    where ω_n is the volume of the unit ball.

    Hint: formula of change of variables.

16. If A, B are $n \times n$ symmetric positive definite matrices, then prove that

    $$\det\left(\frac{A+B}{2}\right) \ge \sqrt{\det A\, \det B},$$

with equality iff $A = B$.

HINT: if O is an orthogonal matrix and $A' = O^tAO$, $B' = O^tBO$, then the inequality holds for A, B iff it holds for A', B'. So we may assume A is diagonal with diagonal $\lambda_1, \cdots, \lambda_n$. Let T be diagonal with diagonal $\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n}$. Show that the inequality holds for A, B iff it holds for $Id, T^{-1}BT^{-1}$. Then assume A is the identity and B is diagonal.

The inequality also follows from Minkowski's inequality in Problem 3 writing

$$\det\left(\frac{A+B}{2}\right) \geq \left((\det(A/2))^{1/n} + (\det(B/2))^{1/n}\right)^n = \frac{1}{2^n}\left((\det A)^{1/n} + (\det B)^{1/n}\right)^n,$$

and using that $x^{1/n} + y^{1/n} \geq 2\sqrt{x^{1/n}y^{1/n}}$.

17. Let K be an open bounded convex set in $\mathbb{R}^n$ (a convex body) with center of mass x_0. Consider all ellipsoids centered at x_0 and containing K, among these there exists an ellipsoid having minimum volume. Prove that this ellipsoid is unique.

    HINT: suppose E_1 and E_2 are two different ellipsoids of minimum volume with corresponding defining matrices A_1 and A_2; we have $|E_1| = |E_2|$ and $A_1 \neq A_2$. Therefore, from Problem 15 $\det A_1 = \det A_2$. Consider the ellipsoid E with corresponding matrix $A = \dfrac{A_1 + A_2}{2}$. Prove that E contains K, and use Problems 15 and 16 to show that $|E| < |E_1|$.

18. We recall the estimate

    $$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} + c_n \operatorname{diam}(\Omega) \left\| \frac{Lu}{(\det A)^{1/n}} \right\|_{L^n(\Omega)} \tag{1}$$

    valid for all $u \in C^2(\Omega) \cap C(\bar{\Omega})$, where $Lu(x) = \operatorname{trace}(A(x)D^2u(x))$ with A symmetric and positive definite in $\bar{\Omega}$.

    Use the John lemma to prove that

    $$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} + c_n |\Omega|^{1/n} \left\| \frac{Lu}{(\det A)^{1/n}} \right\|_{L^n(\Omega)}$$

    whenever Ω is a bounded convex domain.

    HINT: let E be the ellipsoid of minimum volume of Ω, then by John's lemma $\alpha_n E \subset \Omega \subset E$ with α_n constant depending only on n. Let T be an affine transformation such that $T(E) = B_1(0)$, the unit ball, and let $v(z) = u(T^{-1}z)$ defined for $z \in T(\Omega)$. Apply (1) to the function v in $T(\Omega)$.

19. Let Ω be a bounded domain in $\mathbb{R}^n$ and μ is Borel measure over Ω such that $\mu(\Omega) < \infty$. Prove that

(a) for each k there exist a disjoint family $\{\Omega_j^k\}_{j=1}^{N_k}$ of Borel subsets of Ω, such that $\operatorname{diam}(\Omega_j^k) < 1/k$ and $\Omega = \bigcup_{j=1}^{N_k} \Omega_j^k$.

(b) Pick $x_j^k \in \Omega_j^k$ and let $\mu_k = \sum_{j=1}^{N_k} \mu(\Omega_j^k)\, \delta_{x_j^k}$ with $\delta_{x_j^k}$ the Dirac delta concentrated at x_j^k. Prove that $\mu_k \to \mu$ weakly, that is, $\int_\Omega f\, d\mu_k \to \int_\Omega f\, d\mu$ as $k \to \infty$ for each $f \in C(\bar{\Omega})$.

20. Let μ_n and μ be Borel measures in $\Omega \subset \mathbb{R}^n$ that are finite on compact sets (and therefore regular). Suppose that

    (a) $\limsup_{k\to\infty} \mu_k(F) \le \mu(F)$ for each compact $F \subset \Omega$; and

    (b) $\liminf_{k\to\infty} \mu_k(G) \ge \mu(G)$ for each open $G \subset \Omega$.

    Prove that $\mu_k \to \mu$ weakly, that is, $\int_\Omega f(x)\, d\mu_k \to \int_\Omega f(x)\, d\mu$ for all f continuous with compact support in Ω (or for all f continuous and bounded in Ω if $\mu_k(\Omega)$ and $\mu(\Omega)$ are finite).

    HINT: may assume $f \ge 0$. Given $\epsilon > 0$ choose numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_N$ such that $\alpha_0 = 0$, $\sup_\Omega f(x) < \alpha_N$, and $\alpha_j - \alpha_{j-1} = \epsilon$ for $j = 1, \cdots, N$. First prove that

    $$\limsup_{k\to\infty} \int_\Omega f(x)\, d\mu_k \le \int_\Omega f(x)\, d\mu.$$

    Let $F_j = \{x \in \Omega : f(x) \ge \alpha_j\}$, for $j = 0, 1, \cdots, N$; and let $A_j = \{x \in \Omega : \alpha_{j-1} \le f(x) < \alpha_j\} = F_{j-1} \setminus F_j$, $j = 1, \cdots, N$; we have $F_j \subset F_{j-1}$ and $\sum_{j=1}^N \alpha_{j-1} \chi_{A_j}(x) \le f(x) \le \sum_{j=1}^N \alpha_j \chi_{A_j}(x)$. Integrate these inequalities with respect to μ_k, next with respect to μ, use (a), and compare the results. Second prove that

    $$\liminf_{k\to\infty} \int_\Omega f(x)\, d\mu_k \ge \int_\Omega f(x)\, d\mu.$$

    Let $G_j = \{x \in \Omega : f(x) > \alpha_j\}$ and $B_j = G_j \setminus G_{j-1}$; show that $\sum_{j=1}^N \alpha_{j-1} \chi_{B_j}(x) \le f(x) \le \sum_{j=1}^N \alpha_j \chi_{B_j}(x)$. Integrate these inequalities with respect to μ_k, next with respect to μ, use (b), and compare the results.

21. In $\mathbb{R}^3$ consider the function

    $$u(x_1, x_2, x_3) = (1 + x_1^2)(x_2^2 + x_3^2)^{2/3}.$$

    The objective here is to show that u is a convex Aleksandrov solution to $Mu = \phi$ with $\phi \in C^\infty$, $\lambda \le \phi \le \Lambda$ (λ, Λ some positive constants) in a sufficiently small ball around the origin $B_\epsilon(0)$, u on $\partial B_\epsilon(0)$ is continuous, and $u \in C^1(B_\epsilon(0))$ but $u \notin C^2(B_\epsilon(0))$.

    (a) calculate $D^2 u(x_1, x_2, x_3)$ when $(x_1, x_2, x_3) \ne 0$;

(b) show that $\det D^2u(x_1,x_2,x_3) = 32/9\,(1+x_1^2)^2\,((1/3)-(7/3)x_1^2) := \phi(x_1,x_2,x_3)$ for $((x_1,x_2,x_3) \neq 0$;

(c) the function u is not convex in a domain sufficiently far from zero.

(d) calculate the principal minors of D^2u when $(x_1,x_2,x_3) \neq 0$;

(e) prove that for $x_1^2+x_2^2+x_3^2 = \epsilon$ with ϵ sufficiently small, the determinants of the principal minors are positive and therefore $D^2u(x_1,x_2,x_3)$ is positive definite for all $(x_1,x_2,x_3) \neq 0$ in the ball $B_{\sqrt{\epsilon}}(0) = \{x_1^2+x_2^2+x_3^2 \leq \epsilon\}$.
HINT: for the principal minor of order two use Lagrange multipliers.

(f) show that the function u is strictly convex in any convex domain not intersecting the line $\ell \equiv \{x_2 = 0, x_3 = 0\}$ and contained in $B_{\sqrt{\epsilon}}(0)$, and it is C^∞ away from the line ℓ.

(g) the function u is convex in $B_{\sqrt{\epsilon}}(0)$. HINT: use that if a one variable function f is continuous non negative in $[-a,a]$, $f(0) = 0$, f is convex in $(0,a)$ and in $(-a,0)$, then f is convex in $[-a,a]$. Use this as follows. If P_1, P_2 are in $B_{\sqrt{\epsilon}}(0)$, then we need to show the function u restricted to the segment $\overline{P_1P_2}$ is convex as a function of one variable. There are two cases: (a) when the segment $\overline{P_1P_2}$ does not intersect ℓ, then the convexity follows from (f); if $\overline{P_1P_2}$ intersects ℓ, then use the convexity result in one variable.

(h) the graph of u contains the line $x_2 = 0, x_3 = 0$;

(i) $\partial u(\{(x_1,0,0)\})$ has measure zero. HINT: use Aleksandrov's lemma (the set of supporting hyperplanes containing a segment in the graph of u has measure zero).

(j) if $E \subset B_{\sqrt{\epsilon}}(0)$ is a Borel set, then $|\partial u(E)| = \int_E \phi(x)\,dx$. HINT: $\partial u(E) = \partial u(E\cap\ell) \cup \partial u(E\cap\ell^c)$; and from (i) $|\partial u(E\cap\ell)| = 0$; next write E as a disjoint union over the eight octants and since u is C^2 and convex away from ℓ by adding over each piece of E we can represent $|\partial u(E\cap\ell^c)|$ as the integral of ϕ over $E\cap\ell^c$.

(k) $u \in C^{1,1/3}(\partial B_{\sqrt{\epsilon}}(0)$.