**PDES II, Math 8142 (old 562)**
**Prof. Gutiérrez      Homework 3, Sobolev spaces (Due on 3/26/09)**

*Let $\Omega \subset \mathbb{R}^n$ be an open set, $k \geq 0$ an integer and $1 \leq p \leq \infty$. If $f, g \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index we say that $g$ is the $\alpha$-th weak derivative of $f$ if*

$$\int_\Omega g(x)\phi(x)\,dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \phi(x)\,dx,$$

*and we write $g = D^\alpha f$. The Sobolev spaces are defined as follows:*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : \text{the } \alpha\text{-th weak derivative of } f,\ D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq k\},$$

$$\|f\|_{k,p,\Omega} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \approx \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}, \qquad \text{if } p < \infty,$$

*and*

$$\|f\|_{k,\infty,\Omega} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}.$$

*We also have the spaces*

$$H^{k,p}(\Omega) = \overline{\{f \in C^\infty(\Omega) : \|f\|_{k,p,\Omega} < \infty\}};$$

$$W^{k,p}_0(\Omega) = \overline{\{f \in C^\infty_0(\Omega)\}},$$

*where the closures are taken with respect to the norm $\|\cdot\|_{k,p,\Omega}$. We have*

$$W^{k,p}_0(\Omega) \subset H^{k,p}(\Omega) \subset W^{k,p}(\Omega) \subset L^p(\Omega).$$

*If $p < \infty$ then given $f \in W^{k,p}(\Omega)$ there exists a sequence $f_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $f_m \to f$ in the norm $\|\cdot\|_{k,p,\Omega}$. That is*

$$H^{k,p}(\Omega) = W^{k,p}(\Omega), \qquad \text{if } 1 \leq p < \infty.$$

*If $\Omega$ is bounded and has regular boundary, e.g., $\partial\Omega$ is $C^1$, and $1 \leq p < \infty$, then*

(a) *Given $f \in W^{k,p}(\Omega)$ there exists a sequence $f_m \in C^\infty(\overline{\Omega})$ such that $f_m \to f$ in the norm $\|\cdot\|_{k,p,\Omega}$.*

(b) *There exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that $\|Tf\|_{L^p(\Omega)} \leq C\|f\|_{1,p,\Omega}$, with a constant $C > 0$ independent of $f$, and $Tf = f|_{\partial\Omega}$ for $f \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. $Tf$ is the trace of $f$ on $\partial\Omega$.*

1. If $f$ is continuous in $\Omega$ and the weak derivative $D^i f$ is also continuous in $\Omega$, then the ordinary derivative $f_{x_i}$ exists and is continuous in $\Omega$.

   HINT: enough to prove it in dimension one, that is, if

$$\int_a^b f(x) g'(x)\,dx = -\int_a^b h(x) g(x)\,dx \tag{1}$$

with $f$ and $h$ continuous and for all $g \in C_0^\infty(a,b)$, then $f'(x) = h(x)$ in $(a,b)$. Equation (1) is equivalent to

$$\int_a^b \left( f(x) - \int_{x_0}^x h(t)\,dt \right) g'(x)\,dx = 0,$$

so it is enough to show that if

$$\int_a^b w(x)\,g'(x)\,dx = 0, \qquad \forall g \in C_0^\infty(a,b) \tag{2}$$

with $w$ continuous, then $w$ is constant in $(a,b)$. Now, if (2) holds then prove that $\int_a^b w(x)\,g'(x)\,dx = 0$ for all $g$ absolutely continuous with compact support in $(a,b)$ (take $\phi$ smooth with compact support in $[-1,1]$, $\phi_\epsilon(x) = \epsilon^{-1}\phi(x/\epsilon)$ and take $\phi_\epsilon \star g$). Then given $a < a' < b' < b$, define $g(x) = \int_{a'}^x \left( w(t) - \fint_{a'}^{b'} w(s)\,ds \right) dt$, for $a' \le x \le b'$, and $g(x) = 0$ outside $[a', b']$. Show that $g$ is absolutely continuous. Notice that (2) holds if and only if $\int_a^b (w(x) - c)\,g'(x)\,dx = 0$ with any constant $c$. Then from this get that $\int_{a'}^{b'} \left( w(x) - \fint_{a'}^{b'} w(s)\,ds \right)^2 dx = 0$.

2. Let $f(x) = |x_1|$ in the unit ball $B_1(0) \subset \mathbb{R}^n$. Show that $f$ has weak derivatives of order $\alpha$ for all $|\alpha| \le 1$ and $D^1 f = \operatorname{sign} x_1$ and $D^i f = 0$ for $i = 2, \dots, n$.

3. Let $f \in L^1_{loc}(\Omega)$, $\Omega$ connected and $D^\alpha f = 0$ for all $|\alpha| = 1$. Then $f$ is constant in $\Omega$.

4. Assume $\Omega$ is regular. Prove that the trace of a function is unique.

5. Let $\Omega$ be bounded with $C^1$ boundary. Let $f, g \in W^{1,2}(\Omega)$. Prove the formula of integration by parts

$$\int_\Omega D^i f(x) g(x)\,dx = \int_{\partial\Omega} f(x) g(x) \eta_i\,d\sigma(x) - \int_\Omega f(x) D^i g(x)\,dx,$$

where $\eta_i$ is the $i$-th component of the unit outer normal to $\partial\Omega$, and in the surface integral $f$ and $g$ are understood as the traces of $f$ and $g$.

6. Let $x_1, \dots, x_{k+1}$ be numbers and consider the Vandermonde determinant

$$V(x_1, \dots, x_{k+1}) = \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_{k+1} \\ x_1^2 & x_2^2 & \dots & x_{k+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & \dots & x_{k+1}^k \end{bmatrix}.$$

We have that $V(x_1, \dots, x_{k+1}) = \prod_{1 \le i < m \le k+1} (x_m - x_i)$.

Let $B = B_1(0)$ be the unit ball in $\mathbb{R}^n$ and $B^+ = \{x \in B_1(0) : x_n > 0\}$. Let $f \in C^k(\overline{B^+})$, write $x = (x', x_n)$ and define the extension

$$\bar{f}(x) = \begin{cases} f(x), & \text{for } x_n \ge 0 \\ \sum_{i=1}^{k+1} a_i f\left(x', -\dfrac{x_n}{i}\right), & \text{for } x_n < 0, \end{cases}$$

where $a_i, i = 1, \dots, k+1$, is the unique solution of the linear system

$$\sum_{i=1}^{k+1} \left(-\frac{1}{i}\right)^s a_i = 1, \qquad s = 0, \dots, k.$$

Prove that $\bar{f} \in C^k(B)$ and

$$\|\bar{f}\|_{W^{k,p}(B)} \le C \|f\|_{W^{k,p}(B^+)}.$$

7. Let $\lambda > 0$ and $B$ the unit ball in $\mathbb{R}^n$. Show that there exists a constant $C > 0$ depending only on $n$ and $\lambda$ such that if $|\{x \in B : u(x) = 0\}| \ge \lambda$ then

$$\int_B u(x)^2 \, dx \le C \int_B |\nabla u(x)|^2 \, dx.$$

8. Let $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$ and $u \in W^{1,p}(\Omega)$. If $v(x) = f(u(x))$ then $v \in W^{1,p}(\Omega)$ and $Dv = f'(u) Du$.

9. Let $u \in W^{1,p}(\Omega)$ prove that

   (a) $u^+, u^- \in W^{1,p}(\Omega)$, and

$$\nabla u^+ = \begin{cases} \nabla u, & \text{a.e. on } \{u > 0\} \\ 0, & \text{a.e. on } \{u \le 0\}, \end{cases}$$

$$\nabla u^- = \begin{cases} 0, & \text{a.e. on } \{u \ge 0\} \\ \nabla u, & \text{a.e. on } \{u < 0\}, \end{cases}$$

   Hint: consider

$$h_\epsilon(x) = \begin{cases} (x^2 + \epsilon^2)^{1/2} - \epsilon, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0, \end{cases}$$

   and show that $u^+ = \lim_{\epsilon \to 0} h_\epsilon(u)$.

   (b) $\nabla u = 0$ a.e. on $\{u = 0\}$.

   (c) $|u| \in W^{1,p}(\Omega)$.