PDES II, Math 8142 (old 562)
Prof. Gutiérrez Homework 2, Dirichlet integral (Due on 2/24/09)

1. Prove that the Laplacian in polar coordinates is given by

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

2. Let Ω be the unit disc given in polar coordinates r, θ, and suppose $u(x, y)$ is harmonic in Ω with continuous boundary values $f(\theta)$ and assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

   1. Let v be defined by

   $$v(r, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k \left(a_k \cos k\theta + b_k \sin k\theta\right),$$

   where a_k, b_k are the Fourier coefficients of f, that is,

   $$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos k\theta \, d\theta, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin k\theta \, d\theta,$$

   $k = 0, 1, \cdots$.
   Prove that v is well defined for all $0 < r < 1$ and all θ and is a C^2 function for $r < 1$ and is harmonic for $r < 1$.

   2. Prove that

   $$v(r, \theta) = \int_0^{2\pi} f(\phi) P(r, \theta - \phi)\, d\phi, \qquad r < 1,$$

   where $P(r, \xi) = \frac{1}{2\pi}\left(1 + 2\sum_{k=1}^{\infty} r^k \cos k\xi\right)$. HINT: insert the definition of a_k, b_k in the series and interchange the sum and the integral (justify).

   3. Prove using complex exponentials that

   $$P(r, \xi) = \frac{1}{2\pi}\frac{1 - r^2}{1 + r^2 - 2r\cos\xi}, \qquad r < 1.$$

   Notice this is the Poisson kernel for the disc defined in class written in polar coordinates.

   4. The function v has boundary values f. Therefore $u(r\cos\theta, r\sin\theta) = v(r, \theta)$.

   5. Prove that the Dirichlet integral of u written in polar coordinates is

   $$D(u) = \int_0^{2\pi}\int_0^1 (v_\theta^2 + r^2 v_r^2)\frac{1}{r}\, dr d\theta.$$

   HINT: write the gradient of u in polar coordinates.

6. Show that the Dirichlet integral of u

$$D(u) = \int_{\Omega} \left(u_x(x,y)^2 + u_y(x,y)^2 \right) dxdy \approx \sum_{k=1}^{\infty} k(a_k^2 + b_k^2).$$

7. There exist continuous functions f for which the Dirichlet integral for the corresponding u is infinite. HINT: find a_k, b_k for which the series in (6) diverges and the series $\sum_{k=1}^{\infty}(|a_k| + |b_k|)$ converges.