FIRST ORDER PDES

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Contents

These are equations of the form

$$f(x, u, p) = 0,$$

here $x \in \mathbb{R}^n$, $u \in C^1(\mathbb{R}^n)$, and $p = Du$ the gradient of u. They can be solved using the theory of systems of 1st order differential equations and this is a self contained presentation showing how to do it.

1. Quasi-linear equations

For simplicity, we shall first consider $n = 2$ and the case in which the equation is quasi-linear. In this case the pde has the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \tag{1}$$

where the coefficients $a, b, c \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^3$ open. We think that each point $(x, y, z) \in \Omega$ has assigned a vector (a, b, c), i.e.,

$$(x, y, z) \mapsto (a(x, y, z), b(x, y, z), c(x, y, z)), \tag{2}$$

and then the solution $z = u(x, y)$ describes a surface in $\mathbb{R}^3$ whose normal vector $(u_x, u_y, -1)$ is perpendicular to the given vector field. Or in other words, the tangent plane to the surface at each point contains the vector field. The solution to (1) will be find using the so called method of characteristics, that is, finding the solutions of the following system of odes

$$\frac{dx}{dt} = a(x, y, z), \quad \frac{dy}{dt} = b(x, y, z), \quad \frac{dz}{dt} = c(x, y, z). \tag{3}$$

From the theory of odes and since $a, b, c \in C^1(\Omega)$, for each point $(x_0, y_0, z_0) \in \Omega$ there exists a unique solution $(x(t), y(t), z(t))$ to the system (3) defined for $|t| < \epsilon$ and satisfying the initial condition $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$. The idea is to prove that these solutions to the pde (1) are "union" of characteristic curves, that is, solutions to odes (3).

1.1. **Step 1.**

Suppose $z = u(x, y)$ describes a C^1 surface S such that is "union" of characteristic curves, then u solves (1). Indeed, given $(x_0, y_0, z_0) \in S$ there is a characteristic curve C given by $(x(t), y(t), z(t))$ such that $z(t) = u(x(t), y(t))$ and $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$. Then $z'(t) = x'(t)u_x(x(t), y(t)) + y'(t)u_y(x(t), y(t))$, and letting $t = 0$ we get $c(x_0, y_0, z_0) = a(x_0, y_0, z_0)u_x(x_0, y_0) + b(x_0, y_0, z_0)u_y(x_0, y_0)$. That is, for each $(x_0, y_0, z_0) \in S$, u solves (1).

1.2. **Step 2.**

Suppose $z = u(x, y)$ is a C^1 solution to (1) and let S its graph. If $P_0 = (x_0, y_0, z_0) \in S$ and C is a characteristic curve passing through P_0, then $C \subset S$. Indeed, if C has equation $(x(t), y(t), z(t))$ and $(x(0), y(0), z(0)) = P_0$, then we show that $z(t) = u(x(t), y(t))$. Let $U(t) = z(t) - u(x(t), y(t))$, then we have

$$\begin{aligned}\frac{dU}{dt} &= z'(t) - x'(t)\, u_x(x(t), y(t)) - y'(t)\, u_y(x(t), y(t))\\ &= c(x(t), y(t), z(t)) - a(x(t), y(t), z(t))\, u_x(x(t), y(t)) - b(x(t), y(t), z(t))\, u_x(x(t), y(t))\\ &= c\,(x(t), y(t), U(t) + u(x(t), y(t)))\\ &\qquad - a(x(t), y(t), U(t) + u(x(t), y(t)))\, u_x(x(t), y(t))\\ &\qquad - b(x(t), y(t), U(t) + u(x(t), y(t)))\, u_x(x(t), y(t)) := F(U, t).\end{aligned}$$

The function $U = 0$ solves the ode $\frac{dU}{dt} = F(U, t)$ since u solves (1) (notice that $F(U, t)$ depends on the curve and also of u). Also $U(0) = 0$. Since the coefficients a, b, c are C^1, u is also C^1, it follows from the mean value theorem and the form of $F(U, t)$ that $|F(U, t) - F(V, t)| \le C|U - V|$, then by the uniqueness theorem for odes[1] it follows that $U \equiv 0$ and we are done.

1.3. **Cauchy problem.** Given a curve Γ parameterized by $(f(s), g(s), h(s))$ we are looking for a solution u to (1) passing through Γ, that is, $h(s) = u(f(s), g(s))$. Let us assume the curve Γ is C^1, we want to solve (1) in a neighborhood of the point $(x_0, y_0, z_0) = (f(s_0), g(s_0), h(s_0))$. The idea is for each point in $P \in \Gamma$ close to (x_0, y_0, z_0) take the characteristic curve passing through P and then glue all these curves together. This will give the desired solution. More precisely, let $|s - s_0| < \delta$, take $(f(s), g(s), h(s)) \in \Gamma$, and solve (3) with the initial condition $(f(s), g(s), h(s))$. That is, we have a solution to (3), called $(x(s, t), y(s, t), z(s, t))$ with $(x(s, 0), y(s, 0), z(s, 0)) = (f(s), g(s), h(s))$, and let us say this solution is defined for all $|t| < \epsilon$. We then have a transformation

$$\Phi(s, t) = (x(s, t), y(s, t)),$$

defined for $|s - s_0| < \delta$ and $|t| < \epsilon$. The Jacobian matrix of Φ is

$$J_\Phi(s, t) = \begin{bmatrix} \frac{\partial x}{\partial t}(s, t) & \frac{\partial x}{\partial s}(s, t) \\ \frac{\partial y}{\partial t}(s, t) & \frac{\partial y}{\partial s}(s, t) \end{bmatrix}.$$

[1]If $F(U, t)$ is Lipschitz in U (uniformly in t), then the initial value problem $\frac{dU}{dt} = F(U, t)$, $U(0) = U_0$ has at most one solution.

At $s = s_0$ and $t = 0$ we have

$$J_\Phi(s_0,0) = \begin{bmatrix} \dfrac{\partial x}{\partial t}(s_0,0) & \dfrac{\partial x}{\partial s}(s_0,0) \\ \\ \dfrac{\partial y}{\partial t}(s_0,0) & \dfrac{\partial y}{\partial s}(s_0,0) \end{bmatrix} = \begin{bmatrix} a(x_0,y_0,z_0) & f'(s_0) \\ \\ b(x_0,y_0,z_0) & g'(s_0) \end{bmatrix}.$$

If $\det J_\Phi(s_0,0) \neq 0$, then by the inverse function theorem there exists an inverse $\Phi^{-1}(x,y) = (s,t)$ defined locally and we set

$$u(x,y) = z\left(\Phi^{-1}(x,y)\right).$$

Every characteristic curve of (3) is contained in the graph of u, because

$$u(x(s,t), y(s,t)) = z\left(\Phi^{-1}(x(s,t), y(s,t))\right) = z(s,t),$$

so by Subsection 1.1 u solves the pde (1) (this can be also verified analytically). Also from Subsection 1.2 the solution u is unique. Because if u_1 and u_2 are two solutions they both contain any characteristic curve passing by each point in Γ. So we have proved the following theorem.

Theorem 1. *If the coefficients of* (1) *are* C^1 *in a neighborhood of* (x_0,y_0,z_0) *and* Γ *is a* C^1 *curve given by* $(f(s),g(s),h(s))$ *such that* $(f(s_0),g(s_0),h(s_0)) = (x_0,y_0,z_0)$ *and*

$$\det \begin{bmatrix} a(x_0,y_0,z_0) & f'(s_0) \\ \\ b(x_0,y_0,z_0) & g'(s_0) \end{bmatrix} \neq 0, \tag{4}$$

then there exists a unique solution u *to the pde* (1) *defined in a neighborhood of* (x_0,y_0) *such that* u *contains* Γ*, i.e.,* $h(s) = u(f(s),g(s))$ *for* $|s-s_0| < \delta$ *with* δ *sufficiently small.*

2. Degenerate case

This is when

$$\det \begin{bmatrix} a(x_0,y_0,z_0) & f'(s_0) \\ \\ b(x_0,y_0,z_0) & g'(s_0) \end{bmatrix} = 0. \tag{5}$$

We will show that in this case the Cauchy problem might not have solutions.

We shall prove that if there is a solution u to the Cauchy problem then this prescribes the value of the tangent to the initial curve Γ when $s = s_0$, actually, the initial curve Γ will be characteristic at s_0. Indeed, if there is a solution u, then we have $h(s) = u(f(s),g(s))$ for $|s-s_0| < \delta$. Differentiating this with respect to

s yields $h'(s) = f'(s)u_x + g'(s)u_y$ and since u is a solution we also have $c(P_0) = a(P_0)u_x(x_0, y_0) + b(P_0)u_y(x_0, y_0)$. So together with (5) we have the system

$$\begin{aligned} b\,f' - a\,g' &= 0 \\ f'\,u_x + g'\,u_y - h' &= 0 \\ a\,u_x + b\,u_y - c &= 0. \end{aligned}$$

If we make a(second eq)-f'(third eq) and use the first equation we get $-a\,f' + f'\,c = 0$. Also if we make b(second eq)-g′(third eq) and use the first equation we obtain $b\,h' - g'\,c = 0$. Therefore we obtain the equivalent system

$$\begin{aligned} b\,f' - a\,g' &= 0 \\ -c\,f' + a\,h' &= 0 \\ c\,g' + b\,h' &= 0. \end{aligned}$$

The tangent vector to Γ at s_0 is $\tau = (f'(s_0), g'(s_0), h'(s_0))$, and if we set

$$A = \begin{bmatrix} b & -a & 0 \\ -c & 0 & a \\ 0 & -c & b \end{bmatrix},$$

then $A\tau = 0$. If $(a, b, c)\big|_{(x_0,y_0,z_0)}$ is not zero, then rank$(A) = 2$ and therefore the dimension of the space of solutions of $Aw = 0$ is one and so $\tau = \lambda\,(a, b, c)\big|_{(x_0,y_0,z_0)}$. Therefore, if there is a solution u, then the tangent to Γ at s_0 is determined and it must be a multiple of $(a, b, c)\big|_{(x_0,y_0,z_0)}$. Consequently, if (5) holds, and $(f'(s_0), g'(s_0), h'(s_0))$ and $(a, b, c)\big|_{(x_0,y_0,z_0)}$ are not linearly dependent, then there is no solution to the Cauchy problem for the equation (1) with the condition $h(s) = u(f(s), g(s))$.

We also remark that if the initial curve Γ is characteristic, then the Cauchy problem has infinitely many solutions. Indeed, if $(f(s), g(s), h(s))$ is characteristic and passes through (x_0, y_0, z_0) when $s = s_0$, then $f' = a$, $g' = b$ and $h' = c$. Let $\bar{\Gamma} = (\bar{f}(s), \bar{g}(s), \bar{h}(s))$ be defined by $\bar{f}(s) = \alpha(s - s_0) + x_0$, $\bar{g}(s) = \beta(s - s_0) + y_0$ and $\bar{h}(s)$ is any C^1 function such that $\bar{h}(s_0) = z_0$. So

$$\det\begin{bmatrix} a(x_0, y_0, z_0) & \bar{f}'(s_0) \\ b(x_0, y_0, z_0) & \bar{g}'(s_0) \end{bmatrix} = \det\begin{bmatrix} a(x_0, y_0, z_0) & \alpha \\ b(x_0, y_0, z_0) & \beta \end{bmatrix} = a\beta - b\alpha.$$

If $a^2 + b^2 \neq 0$ at (x_0, y_0, z_0), then we can choose α and β such that $a\beta - b\alpha \neq 0$ and so (4) holds and so by Theorem 1, there is a unique solution $\bar{u}$ containing $\bar{\Gamma}$. Since Γ is a characteristic curve, from Subsection 1.2, Γ must be contained in the graph of $\bar{u}$. Therefore, varying α, β or $\bar{h}$ we obtain infinitely many curves $\bar{\Gamma}$ satisfying (4) and so infinitely many solutions $\bar{u}$ to the pde (1) all of them containing Γ.

3. Examples

3.1. **Example 1.** Solve $cu_x + u_y = 0$ with c=constant and satisfying $u(x, 0) = h(x)$ where $h \in C^1$. The curve Γ is given by $(s, 0, h(s))$, condition (4) is then

$$\det \begin{bmatrix} c & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

Solve $x' = c$, $y' = 1$, $z' = 0$ with $(x(s, 0), y(s, 0), z(s, 0)) = (s, 0, h(s))$. We get $x(s, t) = s + ct$, $y(s, t) = t$, and $z(s, t) = h(s)$. Inverting $t = y$ and $s = x - cy$. So the solution is $u(x, y) = h(x - cy)$.

3.2. **Example 2.** Solve $\beta x u_x + u_y = \beta u$ with $c \in \mathbb{R}$ and satisfying $u(x, 0) = h(x)$ where $h \in C^1$. The curve Γ is given by $(s, 0, h(s))$, condition (4) is then

$$\det \begin{bmatrix} \beta x & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

Solve $x' = \beta x$, $y' = 1$, $z' = \beta z$ with $(x(s, 0), y(s, 0), z(s, 0)) = (s, 0, h(s))$. We get $x(s, t) = s\, e^{\beta t}$, $y(s, t) = t$ and $z(s, t) = h(s)\, e^{\beta t}$. Inverting yields $t = y$, $s = x\, e^{-\beta y}$, and so the solution is $u(x, y) = h\left(x\, e^{-\beta y}\right) e^{\beta y}$.

3.3. **Example 3.** Solve $u_x + u_y = u^2$ satisfying $u(x, 0) = h(x)$ where $h \in C^1$. The curve Γ is given by $(s, 0, h(s))$, condition (4) is then

$$\det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

Solve $x' = 1$, $y' = 1$, $z' = z^2$ with $(x(s, 0), y(s, 0), z(s, 0)) = (s, 0, h(s))$. We get $x(s, t) = s + t$, $y(s, t) = t$ and $z(s, t) = -\dfrac{1}{t - 1/h(s)}$. Inverting yields $t = y$, $s = x - y$, and so the solution is $u(x, y) = -\dfrac{1}{y - 1/h(x - y)}$.

3.4. **A maximum principle for a first order pde.** Let $u(x, y) \in C^1\left(\overline{B_1(0)}\right)$ be a solution to the equation

$$a(x, y)u_x + b(x, y)u_y = -u, \qquad \text{in } \overline{B_1(0)}.$$

Suppose that

$$a(x, y)x + b(x, y)y > 0,^2 \qquad \text{for } x^2 + y^2 = 1.$$

Then $u \equiv 0$.

Proof. Let $M = \max_{B_1(0)} u$ and $m = \min_{B_1(0)} u$. We shall prove that $M \leq 0$ and $m \geq 0$. If the maximum is attained in the interior, then from the equation $M = 0$. Similarly, if the minimum is attained in the interior $m = 0$.

Suppose the maximum M is attained at some point (x_0, y_0) on the boundary. If $\nu = (\nu_1, \nu_2)$ is any vector such that

$$\pi/2 < \text{angle}\,(\nu, (x_0, y_0)) \leq \pi. \tag{6}$$

The point $(x_0, y_0)+t\nu \in B_1(0)$ for all $t > 0$ sufficiently small. Let $g(t) = u\left((x_0, y_0) + t\nu\right)$. Since at (x_0, y_0) the maximum of u is attained, then

$$0 \geq g(t) - g(0) = g'(\xi)\, t = \left\{\nu_1 u_x\left((x_0, y_0) + t\nu\right) + \nu_2 u_y\left((x_0, y_0) + t\nu\right)\right\} t,$$

with $0 \leq \xi \leq t$. Since u is C^1 up to the boundary, letting $t \to 0^+$ yields

$$\nu_1 u_x\left(x_0, y_0\right) + \nu_2 u_y\left(x_0, y_0\right) \leq 0. \tag{7}$$

By the assumption $a(x_0, y_0)x_0 + b(x_0, y_0)y_0 = (a(x_0, y_0), b(x_0, y_0)) \cdot (x_0, y_0) > 0$, that is angle $((a(x_0, y_0), b(x_0, y_0)), (x_0, y_0)) < \pi/2$. So $\nu = -(a(x_0, y_0), b(x_0, y_0))$ satisfies (6), and from (7) and the pde we get $u(x_0, y_0) \leq 0$ and we are done. The argument to show $m \geq 0$ is completely similar, in this case the function g satisfies $g(t) - g(0) \geq 0$. □

4. Fully nonlinear case

Very beautiful references for this part are the books by Constantin Carathéodory [Car82, Chapter 3], also containing historical references; the wonderful book by Fritz John [Joh82, pp. 19-31] containing also examples and exercises, and the fundamental treatise by Courant and Hilbert [CH53, Vol.2, pp. 75-103]. Another worthy reference is the book by Sneddon [Sne06, Chapters 1 and 2] for readers interested in finding solutions of many particular 1st order pdes.

[2] This means that the field (a, b) points always towards the outer side of the tangent plane to the boundary of the disc. The result also holds when the disc is replaced by domains having this property.

We consider an equation of the form

$$F(x, y, u, u_x, u_y) = 0, \tag{8}$$

or more generally in any dimensions

$$F(x, u, Du) = 0. \tag{9}$$

We will write as before $p = u_x$ and $q = u_y$. Suppose we have a twice differentiable solution $z = u(x)$ to the pde (9) and set $p_i(x) = u_{x_i}(x)$, $p(x) = D_x u(x)$. Differentiating (9) with respect to x_j yields

$$F_{x_j}(x, u(x), Du(x)) + F_u(x, u(x), Du(x))u_{x_j}(x) + \sum_{i=1}^n F_{p_i}(x, u(x), Du(x))u_{x_i x_j}(x) = 0. \tag{10}$$

Now consider any n-dimensional curve $x = x(t)$, insert it in $u(x)$ and $u_{x_j}(x)$ and let $z(t) = u(x(t))$ and $p_j(t) = u_{x_j}(x(t))$. Differentiating with respect to t yields

$$\dot{z}(t) = \sum_{i=1}^n u_{x_i}(x(t))\dot{x}_i(t) = \sum_{i=1}^n p_i(t)\dot{x}_i(t)$$

and

$$\dot{p}_i(t) = \sum_{j=1}^n u_{x_i x_j}(x(t))\dot{x}_j(t).$$

Suppose now the curve $x(t)$ is chosen so that

$$\dot{x}_i(t) = F_{p_i}(x(t), u(x(t)), Du(x(t))) = F_{p_i}(x(t), z(t), p(t))$$

and substituting this into (10) yields

$$\begin{aligned}
0 &= F_{x_j}(x(t), u(x(t)), Du(x(t))) + F_u(x(t), u(x(t)), Du(x(t)))u_{x_j}(x(t)) \\
&+ \sum_{i=1}^n F_{p_i}(x(t), u(x(t)), Du(x(t)))u_{x_i x_j}(x(t)) \\
&= F_{x_j}(x(t), u(x(t)), Du(x(t))) + F_u(x(t), u(x(t)), Du(x(t)))u_{x_j}(x(t)) \\
&+ \sum_{i=1}^n u_{x_i x_j}(x(t))\dot{x}_i(t) \\
&= F_{x_j}(x(t), u(x(t)), Du(x(t))) + F_u(x(t), u(x(t)), Du(x(t)))u_{x_j}(x(t)) \\
&+ \dot{p}_j(t) \\
&= F_{x_j}(x(t), z(t), p(t)) + F_u(x(t), z(t), p(t))p_j(t) \\
&+ \dot{p}_j(t).
\end{aligned}$$

This means

$$\dot{p}_j(t) = -F_{x_j}(x(t), z(t), p(t)) - F_u(x(t), z(t), p(t))p_j(t).$$

So we obtain the system of $2n + 1$ odes:

$$\dot{x}(t) = D_pF(x(t), z(t), p(t)) \tag{11}$$

$$\dot{p}(t) = -D_xF(x(t), z(t), p(t)) - F_u\,(x(t), z(t), p(t))\; p(t) \tag{12}$$

$$\dot{z}(t) = p(t) \cdot D_pF(x(t), z(t), p(t)). \tag{13}$$

4.1. **Cauchy problem.** Suppose we want to find a solution $z = u(x, y)$ of the pde (8) such that passes through a curve Γ parameterized by $\Gamma(s) = (f(s), g(s), h(s))$. Assume that Γ is C^1 for $|s - s_0| < \delta$ and

$$\Gamma(s_0) = (x_0, y_0, z_0).$$

We want to construct u using the system of odes (11), (12), and (13), which in this particular case is the system of five odes in $(x(t), y(t), z(t), p(t), q(t))$

$$\begin{cases} \dot{x}(t) &= F_p(x, y, z, p, q) \\ \dot{y}(t) &= F_q(x, y, z, p, q) \\ \dot{z}(t) &= pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q) \\ \dot{p}(t) &= -F_x(x, y, z, p, q) - p\,F_z\,(x, y, z, p, q) \\ \dot{q}(t) &= -F_y(x, y, z, p, q) - q\,F_z\,(x, y, z, p, q)\,. \end{cases} \tag{14}$$

As in the quasilinear case we want to find a curve $(x(s, t), y(s, t), z(s, t))$ satisfying the system of five odes and such that $(x(s, 0), y(s, 0), z(s, 0)) = (f(s), g(s), h(s))$. But in this case we have two more unknowns $p(s, t)$ and $q(s, t)$ and so to solve the system of odes we need to prescribe two additional conditions $p(s, 0) = \phi(s)$ and $q(s, 0) = \psi(s)$. These conditions ϕ and ψ must be compatible with the first order pde and also with Γ. In fact, since the prospective solution $z = u(x, y)$ passes through Γ we must have $h(s) = u(f(s), g(s))$ and so differentiating with respect to s

$$h'(s) = u_x(f(s), g(s))f'(s) + u_y(f(s), g(s))g'(s).$$

In addition, the curve must satisfy the pde $F(x, y, z, p, q) = 0$, that is,

$$F(f(s), g(s), h(s), u_x(f(s), g(s)), u_y(f(s), g(s))) = 0.$$

So we need the following compatibility conditions for ϕ and ψ[3]

$$\begin{aligned} h'(s) &= \phi(s)f'(s) + \psi(s)g'(s) \\ F(f(s), g(s), h(s), \phi(s), \psi(s)) &= 0. \end{aligned} \tag{15}$$

In general, there might not be solutions to system of equations, and if there are solutions they might not be unique. So we assume that there exist p_0, q_0 such that

$$\begin{aligned} h'(s_0) &= p_0 f'(s_0) + q_0 g'(s_0) \\ F(x_0, y_0, z_0, p_0, q_0) &= 0, \end{aligned}$$

and

$$\det \begin{bmatrix} F_p(x_0, y_0, z_0, p_0, q_0) & f'(s_0) \\ F_q(x_0, y_0, z_0, p_0, q_0) & g'(s_0) \end{bmatrix} \neq 0. \tag{16}$$

Let $H(s,p,q) = (H_1(s,p,q), H_2(s,p,q)) = (pf'(s) + qg'(s) - h'(s), F(f(s), g(s), h(s), p, q))$ which is defined for $|s - s_0| < \delta$ and for (p,q) close to (p_0, q_0). We have $H(s_0, p_0, q_0) = 0$ and the determinant of the Jacobian matrix $\dfrac{\partial(H_1, H_2)}{\partial(p,q)}$ is different from zero at (s_0, p_0, q_0). Then by the implicit function theorem there exist unique C^1 functions $p = \phi(s)$, $q = \psi(s)$ such that $H(s, \phi(s), \psi(s)) = 0$ for $|s - s_0| < \delta$ with δ sufficiently small. This means that the system (15) can be uniquely solved. Therefore with this choice of $\phi(s)$ and $\psi(s)$ let

$$x(s,t), y(s,t), z(s,t), p(s,t), q(s,t)$$

be the solution to the system (14) satisfying

$$(x(s,0), y(s,0), z(s,0), p(s,0), q(s,0)) = \left(f(s), g(s), h(s), \phi(s), \psi(s)\right).$$

We first observe that

$$G(s,t) := F\left(x(s,t), y(s,t), z(s,t), p(s,t), q(s,t)\right) = 0. \tag{17}$$

[3]Notice the first equation means that the tangent vector $(f'(s), g'(s), h'(s))$ is perpendicular to the vector $(\phi(s), \psi(s), -1)$. That is, at a point $(f(s), g(s), h(s))$ on the curve, the plane through this point with normal $(\phi(s), \psi(s), -1)$ is tangent to the curve. When s moves this tangent plane also moves and describes a ribbon or strip. It is for this reason that a set of functions $f(s), g(s), h(s), \phi(s), \psi(s)$ satisfying the first equation is called a *strip.*

Indeed, from the second equation in (15) $G(s,0) = 0$. Differentiating $G(s,t)$ with respect to t and using the system (14) yields

$$\begin{aligned}\frac{\partial G}{\partial t} =& F_x\dot{x} + F_y\dot{y} + F_z\dot{z} + F_p\dot{p} + F_q\dot{q}\\ &= F_xF_p + F_yF_q + F_z\left(pF_p + qF_q\right) + F_p\left(-F_x - pF_z\right) + F_q\left(-F_y - qF_z\right) = 0,\end{aligned}$$

and so (17) follows from the fundamental theorem of calculus.

Next and with the aid of $x(s,t), y(s,t), z(s,t), p(s,t), q(s,t)$ we construct the solution $z = u(x,y)$ to the 1st order pde. Let us first invert $x(s,t), y(s,t)$. We have $x(s_0,0) = f(s_0)$ and $y(s_0,0) = g(s_0)$. Also the Jacobian of the transformation $\Phi(s,t) = (x(s,t), y(s,t))$ is $\dfrac{\partial(x,y)}{\partial(s,t)} = \begin{bmatrix} x_s(s,t) & x_t(s,t) \\ y_s(s,t) & y_t(s,t)\end{bmatrix}$. If $t = 0$ and $s = s_0$, then $\begin{bmatrix} x_s(s_0,0) & x_t(s_0,0) \\ y_s(s_0,0) & y_t(s_0,0)\end{bmatrix} = \begin{bmatrix} f'(s_0) & F_p(x_0,y_0,z_0,p_0,q_0) \\ g'(s_0) & F_q(x_0,y_0,z_0,p_0,q_0)\end{bmatrix}$ which from assumption (16) has determinant different from zero. Therefore from the inverse function theorem there is a neighborhood of $(s_0,0)$ such that Φ can be inverted, that is, we can write $(s,t) = \Phi^{-1}(x,y)$ with Φ^{-1} a C^1 transformation.

We claim that the desired solution of the 1st order pde is

$$u(x,y) = z(\Phi^{-1}(x,y)). \tag{18}$$

From (17) we have

$$\begin{aligned} 0 &= F\left(x(s,t), y(s,t), z(s,t), p(s,t), q(s,t)\right)\\ &= F\left(x(\Phi^{-1}(x,y)), y(\Phi^{-1}(x,y)), z(\Phi^{-1}(x,y)), p(\Phi^{-1}(x,y)), q(\Phi^{-1}(x,y))\right)\\ &= F\left(x, y, u(x,y), p(\Phi^{-1}(x,y)), q(\Phi^{-1}(x,y))\right),\end{aligned}$$

so if we prove that

$$p(\Phi^{-1}(x,y)) = u_x(x,y), \qquad q(\Phi^{-1}(x,y)) = u_y(x,y),$$

then u solves the 1st order pde. Or equivalently we need to show that

$$p(s,t) = u_x(x(s,t), y(s,t)), \qquad q(s,t) = u_y(x(s,t), y(s,t)), \tag{19}$$

for $|s - s_0| < \delta$ and $|t| < \epsilon$. From (18) we have

$$z(s,t) = u(x(s,t), y(s,t)),$$

and differentiating we get

$$\begin{cases} \dfrac{\partial z}{\partial t}(s,t) = u_x(x(s,t),y(s,t))\dfrac{\partial x}{\partial t}(s,t) + u_y(x(s,t),y(s,t))\dfrac{\partial y}{\partial t}(s,t) \\ \dfrac{\partial z}{\partial s}(s,t) = u_x(x(s,t),y(s,t))\dfrac{\partial x}{\partial s}(s,t) + u_y(x(s,t),y(s,t))\dfrac{\partial y}{\partial s}(s,t). \end{cases} \tag{20}$$

We claim that

$$\begin{cases} \dfrac{\partial z}{\partial t}(s,t) = p(s,t)\dfrac{\partial x}{\partial t}(s,t) + q(s,t)\dfrac{\partial y}{\partial t}(s,t) \\ \dfrac{\partial z}{\partial s}(s,t) = p(s,t)\dfrac{\partial x}{\partial s}(s,t) + q(s,t)\dfrac{\partial y}{\partial s}(s,t). \end{cases} \tag{21}$$

Suppose the claim is proved. Since by assumption (16), $\det\begin{bmatrix} x_s(s_0,0) & x_t(s_0,0) \\ y_s(s_0,0) & y_t(s_0,0)\end{bmatrix} \neq 0$, the linear system (20) has a unique solution for $|s-s_0|<\delta$ and $|t|<\epsilon$ and therefore (19) follows. So it remains to prove (21). From the first three equations in (14) the first equation in (21) immediately follows. To prove the second equation in (21) let

$$r(s,t) := \frac{\partial z}{\partial s}(s,t) - p(s,t)\frac{\partial x}{\partial s}(s,t) - q(s,t)\frac{\partial y}{\partial s}(s,t).$$

We shall prove that $r \equiv 0$. First notice that $r(s,0) = h'(s) - \phi(s)f'(s) + \psi(s)g'(s) = 0$ by (15). On the other hand,

$$\begin{aligned} \frac{\partial r}{\partial t} &= z_{st} - p_t x_s - p x_{st} - q_t y_s - q y_{st} \\ &= (z_t - p x_t - q y_t)_s + p_s x_t + q_s y_t - p_t x_s - q_t y_s \\ &= p_s F_p + q_s F_q + x_s\left(F_x + F_z p\right) + y_s\left(F_y + F_z q\right) \\ &= \frac{\partial G}{\partial s} - F_z(z_s - p x_s - q y_s) = -F_z r, \end{aligned}$$

from (14) and (17). So $r(s,t) = r(s,0)\exp\left(-\int_0^t F_z\,dt\right) = 0$ and we are done.

It remains to verify that $h(s) = u(f(s),g(s))$. But $x(s,0) = f(s)$ and $y(s,0) = g(s)$ so $\Phi^{-1}(f(s),g(s)) = (s,0)$. That is, $u(f(s),g(s)) = z\left(\Phi^{-1}(f(s),g(s))\right) = z(s,0) = h(s)$.

4.2. **Uniqueness.** Notice that the solution to the pde $F(x,y,u,u_x,u_y) = 0$ and passing through the curve Γ, that is, $u(f(s),g(s)) = h(s)$ might not be unique. The number of solutions depends of the number of ways we have to complete Γ to a strip $f(s), g(s), h(s), \phi(s), \psi(s)$, that is, the number of solutions ϕ and ψ of the system (15). However once we choose ϕ and ψ the solution u is unique. Notice this amounts to prescribe u_x and u_y on Γ. As an example of non uniqueness of the Cauchy problem we will show that the equation

$$\frac{1}{2}\left(u_x^2 + u_y^2\right) + xu_x + yu_y = u, \qquad u(x,0) = \frac{1}{2}(1 - x^2),$$

has solutions

$$u(x,y) = \pm y + \frac{1}{2}(1 - x^2).$$

In this case we have $F(x,y,z,p,q) = \dfrac{1}{2}(p^2 + q^2) + xp + yq - z$, and so the system (14) becomes

$$\begin{cases} \dot{x}(t) &= x + p \\ \dot{y}(t) &= y + q \\ \dot{z}(t) &= p(x+p) + q(y+q) \\ \dot{p}(t) &= 0 \\ \dot{q}(t) &= 0. \end{cases}.$$

The curve Γ is $(s, 0, \frac{1}{2}(1 - s^2)$ and then the compatibility conditions (15) in this case are

$$-s = \phi(s), \qquad \frac{1}{2}\left(\phi(s)^2 + \psi(s)^2\right) + s\phi(s) - \frac{1}{2}(1 - s^2) = 0,$$

that is,

$$\phi(s) = -s, \qquad \psi(s) = \pm 1.$$

So $p(s,t) = -s$, and $q(s,t) = \pm 1$. Also the Jacobian condition (16) holds because

$$\det \begin{bmatrix} p + x & 1 \\ & \\ q + y & 0 \end{bmatrix} = -(y + q)$$

which at $q = \pm 1$ and $y = 0$ is different from zero. Next we need to solve $\dot{x} = x - s$, $\dot{y} = y \pm s$ with initial conditions $x(s,0) = s$ and $y(s,0) = 0$, which yields $x(s,t) = s$ and $y(s,t) = \pm(e^t - 1)$. Then $\dot{z} = e^t$ and so $z(s,t) = e^t - \frac{1}{2} - \frac{1}{2}s^2$. Therefore $s = x$ and $\pm y + 1 = e^t$. So

$$u(x,y) = \pm y + 1 - \frac{1}{2} - \frac{1}{2}x^2 = \pm y + \frac{1}{2}(1 - x^2).$$

4.3. **Solution of the eikonal equation.** The eikonal equation in dimension two when the index of refraction is constant is given by

$$c^2\left((u_x)^2+(u_y)^2\right)=1,$$

where c is a constant. Therefore $F(x,y,z,p,q)=\dfrac{1}{2}\left(c^2(p^2+q^2)-1\right)$, (written in this way for convenience in the calculations). The system (14) then becomes

$$\begin{cases}\dot{x}(t) &= c^2 p\\ \dot{y}(t) &= c^2 q\\ \dot{z}(t) &= c^2(p^2+q^2)\\ \dot{p}(t) &= 0\\ \dot{q}(t) &= 0.\end{cases}.$$

Let us fix an initial curve $\Gamma=(f(s),g(s),h(s))$ and let us complete it to a strip by adding $\phi(s)$ and $\psi(s)$. Then the compatibility conditions (15) become

$$h'(s)=\phi(s)f'(s)+\psi(s)g'(s),\qquad \phi(s)^2+\psi(s)^2=c^{-2}.$$

This imply that $h'(s)^2=\left((\phi(s),\psi(s))\cdot(f'(s),g'(s))\right)^2\leq|(\phi(s),\psi(s))|^2\,|(f'(s),g'(s))|^2\leq c^{-2}\left(f'(s)^2+g'(s)^2\right)$. That is, there are no real solutions ϕ and ψ if

$$h'(s)^2>c^{-2}\left(f'(s)^2+g'(s)^2\right).$$

So if $h'(s)^2\leq c^{-2}\left(f'(s)^2+g'(s)^2\right)$, then there are two pairs of solutions $\phi(s),\psi(s)$, (the line intersects the circle $a^2+b^2=c^{-2}$ in two points).

Suppose the curve Γ lies on the plane x,y, that is, $\Gamma=(f(s),g(s),0)$. Clearly in this case, the compatibility condition has two pairs of solutions (ϕ,ψ) and $(-\phi,-\psi)$. Solving the system of odes yields

$$\begin{aligned}&x(s,t)=f(s)+c^2t\phi(s),\qquad y(s,t)=g(s)+c^2t\psi(s),\qquad z(s,t)=t,\\ &p(s,t)=\phi(s),\qquad q(s,t)=\psi(s).\end{aligned}$$

Condition (16) reads in this case

$$\det\begin{bmatrix}c^2\phi(s) & f'(s)\\ c^2\psi(s) & g'(s)\end{bmatrix}=c^2\left(\phi(s)g'(s)-\psi(s)f'(s)\right).$$

From the compatibility conditions $0=\phi(s)f'(s)+\psi(s)g'(s)$, $\phi(s)^2+\psi(s)^2=c^{-2}$. If we assume that $f'(s)^2+g'(s)^2>0$, then $\phi(s)g'(s)-\psi(s)f'(s)\neq 0$. Therefore the map $(s,t)\mapsto(f(s)+c^2t\phi(s),g(s)+c^2t\psi(s))$ is invertible and the solution u can be found with the inverse Φ^{-1} of this map setting $u(x,y)=z(\Phi^{-1}(x,y))$.

4.4. **Higher dimensional case.** Now with the same method we can solve the equation

$$F(x, u, Du(x)) = 0.$$

We write $p = Du(x)$ and $z = u$ and the Cauchy problem now is to find an n-dimensional surface $z = u(x)$ passing through an $(n-1)$-dimensional manifold Γ parameterized by

$$\Gamma(s_1, \cdots, s_{n-1}) = (f_1(s_1, \cdots, s_{n-1}), \cdots, f_n(s_1, \cdots, s_{n-1}), h(s_1, \cdots, s_{n-1}))$$

with $\Gamma(s_1^0, \cdots, s_{n-1}^0) = (x_1^0, \cdots, x_n^0, z^0) = P_0$ (we set $s_0 = (s_1^0, \cdots, s_{n-1}^0)$, $s = (s_1, \cdots, s_{n-1})$, and $x_0 = (x_1^0, \cdots, x_n^0)$). We then need to construct n-functions

$$\phi_1(s_1, \cdots, s_{n-1}), \cdots, \phi_n(s_1, \cdots, s_{n-1}),$$

compatible with the curve Γ and the equation. That is, in analogy with (15), we need

$$\frac{\partial h}{\partial s_i} = \sum_{j=1}^{n} \phi_j \frac{\partial f_j}{\partial s_i}, \qquad i = 1, \cdots, n;$$

and

$$F(f_1, \cdots, f_n, h, \phi_1, \cdots, \phi_n) = 0.$$

To solve this system of equation we assume as before the existence of a point $p_0 = (p_1^0, \cdots, p_n^0)$ such that

$$\frac{\partial h}{\partial s_i}(s_0) = \sum_{j=1}^{n} p_j^0 \frac{\partial f_j}{\partial s_i}(s_0), \qquad i = 1, \cdots, n;$$

and

$$F(f_1(s_0), \cdots, f_n(s_0), h(s_0), p_1^0, \cdots, p_n^0) = 0,$$

and in analogy with condition (16), we also assume

$$\det \begin{bmatrix} F_{p_1}(x_0, , z^0, p_0) & \frac{\partial f_1}{\partial s_1}(s_0) & \cdots & \frac{\partial f_1}{\partial s_{n-1}}(s_0) \\ \vdots & \vdots & & \vdots \\ F_{p_n}(x_0, , z^0, p_0) & \frac{\partial f_n}{\partial s_1}(s_0) & \cdots & \frac{\partial f_n}{\partial s_{n-1}}(s_0) \end{bmatrix} \neq 0.$$

Then as in the three dimensional case, by the implicit function theorem we obtain the desired functions $\phi_1(s_1, \cdots, s_{n-1}), \cdots, \phi_n(s_1, \cdots, s_{n-1})$. With these functions as initial conditions, we solve the system of odes (11), (12) and (13) obtaining the solutions

$$x(s, t), z(s, t), p(s, t)$$

such that $(x(s,0), z(s,0), p(s,0)) = (f_1(s), \cdots, f_n(s), h(s), \phi(s), \cdots, \phi_n(s))$. Now the map $(s,t) \mapsto x = x(s,t)$ has an inverse $\Phi^{-1}(x) = (s,t)$ and the solution u is given by

$$u(x) = z(\Phi^{-1}(x)).$$

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