PDES, Math 8142 (old 562), Prof. Gutiérrez, Final Exam (Due on 05/12/09, 11:30AM, room 527)

1. Prove that all solutions of the pde
$$xu_x - yu_y = 0$$
have the form $u(x,y) = f(xy)$.
HINT: make the change of variables $y = r/x$.

2. Let $h \in C^1(\mathbb{R})$. Prove that the solution of the Cauchy problem
$$u\, u_x + u_y = 0, \qquad u(x,0) = h(x),$$
satisfies $u(x,y) = h(x - u(x,y)y)$.

3. Solve the Cauchy problem
$$x^2\, u_x + y^2\, u_y = u^2, \qquad u(x,2x) = 1.$$

4. Let $n > 1$. Prove that the function $|x|^{-\alpha} \in W^{1,p}(B_1(0))$ if and only if $\alpha < \dfrac{n-p}{p}$.

5. Verify that if $n > 1$ the function $u(x) = \log\log\left(1 + \dfrac{1}{|x|}\right)$ belongs to $W^{1,n}(B_1(0))$ and is unbounded.

6. Let $f_m \in C^k(\Omega)$ be a sequence of functions such that $f_m \to f$ weakly in $L^2(\Omega)$. Suppose $\|D^\alpha f_m\|_{L^2(\Omega)} \le M$ for all m and $|\alpha| = k$. Prove that the weak derivative $D^\alpha f$ exists and belongs to $L^2(\Omega)$.

7. Suppose Ω is a connected domain. A function $u \in W^{1,2}(\Omega)$ is a weak solution of the Neumann problem
$$-\Delta u = f \text{ in } \Omega, \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1}$$
if
$$\int_\Omega Du \cdot Dv\, dx = \int_\Omega f v\, dx$$
for all $v \in W^{1,2}(\Omega)$.
Prove that (1) has a weak solution if and only if $\int_\Omega f\, dx = 0$.

8. Let u be a smooth solution of $Lu = -\sum_{i,j=1}^n a_{ij}u_{ij} = 0$ in Ω, where a_{ij} is uniformly elliptic in Ω. Let $v = |Du|^2 + \lambda u^2$. Prove that $Lv \le 0$ in Ω for λ sufficiently large. Conclude from the maximum principle that
$$\|Du\|_{L^\infty(\Omega)} \le C\left(\|Du\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\right).$$

9. Show that the function $e^{-|x|^2}$ has an absolute maximum at $x = 0$ and solves the equation $\Delta u + (2n - 4|x|^2)u = 0$. What is the implication of this example for the validity of the maximum principle for the operator $\Delta u + cu$?