SOLUTION OF THE DIRICHLET PROBLEM WITH A VARIATIONAL METHOD

CRISTIAN E. GUTIÉRREZ
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1. Dirichlet integral

Let $f \in C(\partial\Omega)$ with Ω open and bounded. Let $H = \{u \in C^1(\bar{\Omega}) : u = f \text{ on } \partial\Omega\}$ and

$$D(u) = \int_\Omega |Du(x)|^2 \, dx.$$

The objective is to prove that with minimizers of $D(u)$ over H one can solve the Dirichlet problem in Ω.

We assume that f satisfies the following property: there exists $v \in C^1(\bar{\Omega})$ such that $v = f$ on $\partial\Omega$. This is not a restriction to solve the Dirichlet problem with this approach because if $f \in C(\partial\Omega)$ then by the Wierstrass approximation theorem there exist polynomials f_k in $\mathbb{R}^n$ such that $f_k|_{\partial\Omega}$ converge uniformly to f on $\partial\Omega$. If we can solve the Dirichlet problem with data $u_k = f_k$ on $\partial\Omega$, then by the maximum principle $u_k \to u$ uniformly in Ω for some u and therefore u is harmonic in Ω and has boundary values f.

If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies $\Delta u = 0$ in Ω and $u = f$ on $\partial\Omega$, then

$$D(u) \leq D(v), \qquad \forall v \in H.$$

That is, the Dirichlet integral is minimized by the solution of the Dirichlet problem. This follows writing $g = v - u$ with $v \in H$

$$\begin{aligned} D(v) &= D(g+u) = D(g) + 2D(g,u) + D(u) \\ &= D(g) + D(u) + 2\int_{\partial\Omega} g D_\nu u \, d\sigma - 2\int_\Omega g \Delta u \, dx \\ &= D(g) + D(u) \geq D(u) \end{aligned}$$

from the first Green formula. There are continuous functions $f \in C(\partial\Omega)$ such that the solution u of the Dirichlet problem with data f satisfies $D(u) = +\infty$. An example is the function $u(r,\theta) = \sum_{k=1}^{\infty} \dfrac{r^{n!}\cos(n!\theta)}{n^2}$ that is harmonic in the unit disk, it has boundary values $f(\theta) = \sum_{k=1}^{\infty} \dfrac{\cos(n!\theta)}{n^2}$, and $D(u) = +\infty$.

We have that $D(\lambda u + v) = \lambda^2 D(u) + 2\lambda D(u,v) + D(v) \geq 0$ for all λ, and so $D(u)D(v) - D(u,v) \geq 0$. Therefore

$$\|u\|_D = D(u)^{1/2}$$

defines a quasi norm in H, that is, $\|\cdot\|_D$ satisfies the triangle inequality and $\|\lambda u\|_D = |\lambda|\|u\|_D$.

2. Poincaré inequality

Let $H = \{u \in C^1(\bar{\Omega}) : u = f \text{ on } \partial\Omega\}$ and $H_0 = \{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\}$.

Lemma 1. *There exists a constant $C > 0$, depending only on the domain Ω, such that*

$$\int_\Omega w(x)^2\,dx \leq C \int_\Omega |Dw(x)|^2\,dx,$$

for all $w \in H_0$.

Proof. Since Ω is bounded, $\Omega \subset Q = [-a,a] \times [-a,a]$. Let $\bar{w}(x,y) = w(x,y)$ for $(x,y) \in \Omega$ and $\bar{w}(x,y) = 0$ for $(x,y) \in Q \setminus \Omega$. We assume that the intersection of Ω with each vertical line is finite union of open intervals. Then the function $\bar{w}(x,\cdot)$ is Lipschitz and therefore absolutely continuous. So we can write

$$\bar{w}(x,y) = \int_{-a}^{y} \bar{w}_y(x,\xi)\,d\xi.$$

Then squaring and using Cauchy-Schwartz we get

$$\bar{w}(x,y)^2 \leq (y+a) \int_{-a}^{y} \bar{w}_y(x,\xi)^2\,d\xi \leq 2a \int_{-a}^{a} \bar{w}_y(x,\xi)^2\,d\xi,$$

and integrating this inequality in x yields

$$\int_{-a}^{a} \bar{w}(x,y)^2\,dx \leq 2a \int_{-a}^{a}\int_{-a}^{a} \bar{w}_y(x,\xi)^2\,d\xi\,dx.$$

Now integrating in y yields

$$\int_{-a}^{a}\int_{-a}^{a} \bar{w}(x,y)^2\,dxdy \leq 4a^2 \int_{-a}^{a}\int_{-a}^{a} \bar{w}_y(x,\xi)^2\,d\xi\,dx.$$

From the definition of $\bar{w}$ and since $\bar{w}_y(x,y) = 0$ when $(x,y) \in Q \setminus \bar{\Omega}$, the lemma then follows with $C = 4a^2$. □

Remark 2. It is clear that if $\Omega \subset [a,b] \times [c,d]$, then the estimate holds with $C = (b-a)(d-c)$. In higher dimensions, one obtains in the same way that if $\Omega \subset R$ with R an n-dimensional interval, then the lemma holds with $C = |R|$.

3. Solution to the Dirichlet problem

By our assumption the set $H \neq \emptyset$, and therefore $\inf_H D(u) = L < \infty$. So there exists a sequence (possibly not unique) $v_k \in H$ such that $D(v_k) \to L$ as $k \to \infty$. We call v_k a minimizing sequence, and the objective is to construct with v_k a harmonic function ϕ in Ω such that $\phi = f$ on $\partial\Omega$.

Lemma 3. *For each $w \in H_0$, we have*

$$\lim_{k\to\infty} D(v_k, w) = 0,$$

where v_k is the minimizing sequence. Moreover, if $D(w_k) \leq M$ with $w_k \in H_0$, then $D(v_k, w_k) \to 0$ as $k \to \infty$.

Proof. We have $v_k + \epsilon w \in H$ for all ϵ and so

$$L \leq D(v_k + \epsilon w) = D(v_k) + 2\epsilon D(v_k, w) + \epsilon^2 D(w),$$

and the minimum of the right hand side is attained when $\epsilon = -\dfrac{D(v_k, w)}{D(w)}$ which yields

$$L \leq D(v_k) - \frac{D^2(v_k, w)}{D(w)}.$$

Therefore

$$|D(v_k, w)| \leq (D(v_k) - L)^{1/2}\ D(w)^{1/2},$$

and the lemma follows. □

Lemma 4. *The minimizing sequence v_k is a Cauchy sequence in the norm $\|\cdot\|_{L^2(\Omega)} + \|\cdot\|_D$.*

Proof. Let $w = v_k - v_m$. We have $D(v_k) = D(v_m + w) = D(v_m) + 2D(v_m, w) + D(w)$. So

$$|D(w)| \leq |D(v_k) - D(v_m)| + 2|D(v_m, w)|$$

and therefore from the previous lemma, $\|w\|_D \to 0$ as $k, m \to \infty$. From the Poincaré inequality, $\|w\|_{L^2(\Omega)} \leq C\|w\|_D$ and the lemma follows. □

Given $x \in \Omega$ and $B_\rho(x) \subset \Omega$ we let

$$\phi_k(x, \rho) = \fint_{B_\rho(x)} v_k(z)\, dz.$$

Lemma 5. *Let K be closed $K \subset \Omega$, and fix $\rho < dist(K, \Omega^c)$. Then $\phi_k(x, \rho)$ are continuous and converge uniformly for $x \in K$ to a function $\phi(x, \rho)$ for each ρ.*

Proof. The functions $\phi_k(x, \rho)$ are clearly continuous. We write

$$\begin{aligned}|\phi_k(x, \rho) - \phi_m(x, \rho)| &= \left| \fint_{B_\rho(x)} (v_k(z) - v_m(z))\, dz \right| \\ &\leq \left(\fint_{B_\rho(x)} |v_k(z) - v_m(z)|^2\, dz \right)^{1/2} \leq \frac{1}{|B_\rho(x)|^{1/2}} \|v_k - v_m\|_{L^2(\Omega)},\end{aligned}$$

so the sequence $\phi_k(x, \rho)$ is uniformly Cauchy and the lemma follows. □

Lemma 6. *The function $\phi(x, \rho)$ is independent of ρ, i.e., $\phi(x, \rho) = \phi(x)$.*

Proof. Fix $x \in \Omega$ and let $\rho_1 < \rho_2$ with $B_{\rho_i}(x) \subset \Omega$. Suppose we are in dimension two and let

$$w(x) = \begin{cases} \dfrac{1}{2\pi} \left(\log \dfrac{\rho_1}{\rho_2} + \dfrac{1}{2} |x - z|^2 \left(\dfrac{1}{\rho_1^2} - \dfrac{1}{\rho_2^2} \right) \right), & |x - z| \leq \rho_1 \\ \dfrac{1}{2\pi} \left(\log \dfrac{|x - z|}{\rho_2} + \dfrac{1}{2} \left(1 - \dfrac{|x - z|^2}{\rho_2^2} \right) \right), & \rho_1 < |x - z| \leq \rho_2 \\ 0 & \rho_2 < |x - z|. \end{cases}$$

We have

$$Dw(x) = \begin{cases} \dfrac{1}{2\pi} (x - z) \left(\dfrac{1}{\rho_2^2} - \dfrac{1}{\rho_1^2} \right), & |x - z| \leq \rho_1 \\ \dfrac{1}{2\pi} (x - z) \left(\dfrac{1}{\rho_2^2} - \dfrac{1}{|x - z|^2} \right), & \rho_1 < |x - z| \leq \rho_2 \\ 0 & \rho_2 < |x - z|. \end{cases}$$

We have that $w \in H_0$ and

$$\Delta w(x) = \begin{cases} \dfrac{1}{\pi \rho_1^2} - \dfrac{1}{\pi \rho_2^2}, & |x - z| < \rho_1 \\ -\dfrac{1}{\pi \rho_2^2}, & \rho_1 < |x - z| < \rho_2 \\ 0 & \rho_2 < |x - z|. \end{cases}$$

To apply the first Green formula we remove the places where the Laplacian of w is discontinuous. We set

$$\Omega_\epsilon = \Omega \setminus \left(\{z : \rho_1 - \epsilon < |x - z| < \rho_1 + \epsilon\} \cup \{z : \rho_2 - \epsilon < |x - z| < \rho_2 + \epsilon\} \right).$$

By the first Green formula applied in Ω_ϵ we have

$$\int_{\Omega_\epsilon} Dv_k \cdot Dw\, dx + \int_{\Omega_\epsilon} v_k\, \Delta w\, dx = \int_{\partial\Omega_\epsilon} v_k(z)\, \partial_\eta w(x)\, d\sigma(z).$$

From the form of the gradient of w, the right hand side of the last identity tends to zero as $\epsilon \to 0$, and so we obtain

$$0 = D(v_k, w) + \int_\Omega v_k \Delta w\, dx = D(v_k, w) + \fint_{B_{\rho_1}(x)} v_k(z)\, dz - \fint_{B_{\rho_2}(x)} v_k(z)\, dz,$$

and letting $k \to \infty$ the lemma follows. □

Theorem 7. *The function ϕ is harmonic in Ω and $\phi = f$ on $\partial\Omega$.*

Proof. We claim that if $B_a(x_0), B_b(x_0) \subset \Omega$, then

$$\fint_{B_a(x_0)} \phi_k(x, a)\, dx = \fint_{B_b(x_0)} \phi_k(x, b)\, dx.$$

In fact, we have

$$\phi_k(x, a) = \fint_{B_a(x)} v_k(z)\, dz = \fint_{B_a(0)} v_k(z + x)\, dz,$$

and integrating in x

$$\begin{aligned}\fint_{B_b(x_0)} \phi_k(x, a)\, dx &= \fint_{B_b(x_0)} \fint_{B_a(0)} v_k(z + x) dz\, dx = \fint_{B_a(0)} \fint_{B_b(x_0)} v_k(z + x) dx\, dz \\ &= \fint_{B_a(0)} \fint_{B_b(x_0+z)} v_k(u) du\, dz = \fint_{B_a(0)} \phi_k(x_0 + z, b)\, dz \\ &= \fint_{B_a(x_0)} \phi_k(z, b)\, dz,\end{aligned}$$

which proves the claim. Letting $k \to \infty$ we obtain

$$\fint_{B_a(x_0)} \phi(x)\, dx = \fint_{B_b(x_0)} \phi(x)\, dx,$$

and since ϕ is continuous, letting $b \to 0$ yields

$$\fint_{B_a(x_0)} \phi(x)\, dx = \phi(x_0),$$

that is, ϕ satisfies the mean value property in Ω and the first part of the theorem is then proved.

We shall prove in dimension two that

$$\lim_{x \to x_0, x \in \Omega} \phi(x) = f(x_0), \qquad x_0 \in \partial\Omega,$$

under the following assumption on Ω: there exists $R > 0$ such that $\{z : |z - x| = \rho\} \cap \partial\Omega \neq \emptyset$ for all $x \in \partial\Omega$ and for all $\rho \leq R$. Let $x_0 \in \partial\Omega$ and consider the ball $B_h(x_0)$, and let $x \in B_{h/2}(x_0) \cap \Omega$. Let $\sigma = \operatorname{dist}(x, \partial\Omega)$. There exists $x_1 \in \partial\Omega$ such that $|x_1 - x| = \sigma$. We have $\sigma \leq h/2, |x_0 - x_1| < h$, and $B_{\sigma/2}(x) \subset B_{3\sigma/2}(x_1) \cap \Omega$. We write

$$\begin{aligned}|f(x_0) - \phi(x)| &\leq |f(x_0) - f(x_1)| + |f(x_1) - v_k(x)| \\ &\qquad + \left| v_k(x) - \fint_{B_{\sigma/2}(x)} v_k(z)\,dz \right| + \left| \fint_{B_{\sigma/2}(x)} v_k(z)\,dz - \phi(x) \right| \\ &= I + II + III + IV.\end{aligned}$$

Since f is continuous, I is small for h small. II is small since $v_k \in C(\bar{\Omega})$ and $v_k = f$ on $\partial\Omega$. III is also small for h small since $\sigma \leq h/2$.

We estimate IV. Consider the circle $C_\rho(x_1)$ for $\rho \leq 3\sigma/2$. If $3\sigma/2 < R$, where R appears in the condition on Ω, then $C_\rho(x_1)$ intersects $\partial\Omega$ at some point x_ρ. Let $w = v_k - v_m$. We have $w \in H_0(\Omega)$ and we extend w to be zero outside $\bar{\Omega}^c$ and we still call this extension w. Let $h(\theta) = w(x_1 + \rho(\cos\theta, \sin\theta))$, $x' \in C_\rho(x_1)$, and $x_1 + \rho(\cos\theta_0, \sin\theta_0) = x_\rho, x_1 + \rho(\cos\theta_1, \sin\theta_1) = x'$. We have

$$w(x') = \int_{\theta_0}^{\theta_1} h'(\theta)\,d\theta = \int_{\theta_0}^{\theta_1} Dw(x_1 + \rho(\cos\theta, \sin\theta)) \cdot (-\sin\theta, \cos\theta)\rho\,d\theta.$$

Then

$$\begin{aligned}|w(x')| &\leq \int_{\theta_0}^{\theta_1} |Dw(x_1 + \rho(\cos\theta, \sin\theta))|\rho\,d\theta \\ &\leq \rho|\theta_1 - \theta_0|^{1/2} \left(\int_{\theta_0}^{\theta_1} |Dw(x_1 + \rho(\cos\theta, \sin\theta))|^2\,d\theta \right)^{1/2} \\ &\leq \rho(2\pi)^{1/2} \left(\int_0^{2\pi} |Dw(x_1 + \rho(\cos\theta, \sin\theta))|^2\,d\theta \right)^{1/2},\end{aligned}$$

that is,

$$|w(x_1 + \rho(\cos\theta_1, \sin\theta_1))|^2 \leq 2\pi\,\rho^2 \int_0^{2\pi} |Dw(x_1 + \rho(\cos\theta, \sin\theta))|^2\,d\theta.$$

Integrating this inequality for $0 \leq \theta_1 \leq 2\pi$ yields

$$\int_0^{2\pi} |w(x_1 + \rho(\cos\theta_1, \sin\theta_1))|^2\,d\theta_1 \leq 4\pi^2\,\rho^2 \int_0^{2\pi} |Dw(x_1 + \rho(\cos\theta, \sin\theta))|^2\,d\theta,$$

and now integrating for $0 \leq \rho \leq 3\sigma/2$ we obtain

$$\int_{B_{3\sigma/2}(x_1)} w(z)^2\,dz \leq 9\pi^2\,\sigma^2 \int_{B_{3\sigma/2}(x_1)} |Dw(z)|^2\,dz.$$

Therefore

$$\begin{aligned}\left|\fint_{B_{\sigma/2}(x)} (v_k(z) - v_m(z))\,dz\right| &\leq \left(\fint_{B_{\sigma/2}(x)} |v_k(z) - v_m(z)|^2\,dz\right)^{1/2}\\ &\leq 3\left(\fint_{B_{3\sigma/2}(x_1)} |w(z)|^2\,dz\right)^{1/2}\\ &\leq 2\sqrt{\pi}\left(\int_{B_{3\sigma/2}(x_1)} |Dw(z)|^2\,dz\right)^{1/2}\\ &\leq 2\sqrt{\pi}\left(\int_{\Omega} |D(v_k - v_m)(z)|^2\,dz\right)^{1/2}\end{aligned}$$

which tends to zero as $k, m \to \infty$. Lettting $m \to \infty$ we obtain

$$\left|\fint_{B_{\sigma/2}(x)} v_k(z)\,dz - \phi(x)\right| < \epsilon$$

for all k sufficiently large and so IV is also small.

□

Department of Mathematics, Temple University, Philadelphia, PA 19122
E-mail address: `gutierre@temple.edu`