H=W

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We prove that $C^\infty(\Omega) \cap W^{k,p}(\Omega)$, $1 \le p < \infty$, is dense in $W^{k,p}(\Omega)$ with the norm

$$\|u\|_{k,p} = \sum_{|\alpha| \le k} \|D^\alpha u\|_p.$$

Let Ω_j, $j = 1, 2, \cdots$ be open domains strictly contained in Ω such that $\bar{\Omega}_j \subset \Omega_{j+1}$ and $\Omega = \bigcup_{j=1}^\infty \Omega_j$.

Let ϕ_j be a partition of unity subordinated to the family of open sets $U_j = \Omega_{j+1} \setminus \bar{\Omega}_{j-1}$, with $\Omega_0 = \Omega_{-1} = \emptyset$, $j = 0, 1, \cdots$; $\Omega = \bigcup_{j=0}^\infty U_j$. That is,

(1) $\phi_j \in C_0^\infty(U_j)$, $0 \le \phi_j \le 1$;
(2) For each $y \in \Omega$ there exists an open ball $B_r(y) \subset \Omega$ such that the set $\{j : \text{supp}(\phi_j) \cap B_r(y) \ne \emptyset\}$ finite;
(3) $\sum_{j=0}^\infty \phi_j(x) = 1$ for all $x \in \Omega$.

Let $u \in W^{k,p}(\Omega)$, $\eta \in C_0^\infty(B_1(0))$, $0 \le \eta \le 1$, and $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$. We have $\phi_j u \in W^{k,p}(\Omega)$, $\eta_\epsilon \star (\phi_j u) \to \phi_j u$ as $\epsilon \to 0$ in $W^{k,p}(\Omega)$. Let $\epsilon > 0$ and fix $0 \le j < \infty$. There exists h_j, $\dfrac{1}{j+1} \ge h_j > 0$ such that

(1) $\|\eta_{h_j} \star (\phi_j u) - \phi_j u\|_{k,p} \le \dfrac{\epsilon}{2^j}$;
(2) $h_j \le \text{dist}(\Omega_j, \partial\Omega_{j+1})$, $j \ge 1$.

The functions $v_j = \eta_{h_j} \star (\phi_j u)$, $j = 1, 2, \cdots$ satisfy that for each $y \in \Omega$, the set

$$\{j : \text{supp}(v_j) \cap B_{r/2}(y) \ne \emptyset\}$$

is finite. Indeed, since $\text{supp}(v_j) \subset \text{supp}(\eta_{h_j}) + \text{supp}(\phi_j) = B_{h_j}(0) + \text{supp}(\phi_j)$, if $z \in \text{supp}(v_j) \cap B_{r/2}(y)$, then $z = z_j + \delta_j$ with $z_j \in \text{supp}(\phi_j)$, $|\delta_j| < h_j$ and $|z - y| < r/2$. So $|z_j - y| < \dfrac{r}{2} + h_j < r$ for j large, that is, $\text{supp}(\phi_j) \cap B_r(y) \ne \emptyset$ for j large and the conclusion follows from item (2) in the definition of partition of unity. Therefore given any open set Ω' with compact closure contained in Ω, we have that $\{j : \text{supp}(v_j) \cap \Omega' \ne \emptyset\}$ is a finite set. Therefore the function $v(x) = \sum_{j=0}^\infty v_j(x)$ has only a finite number of non vanishing terms in its sum for all $x \in \Omega'$ and therefore $v \in C^\infty(\Omega)$.

We have

$$\begin{aligned}
\|u-v\|_{k,p} &= \left\|\sum_{j=0}^{\infty}(\phi_j u - v_j)\right\|_{k,p} \\
&= \sum_{|\alpha|\le k}\left(\int_\Omega \left|D^\alpha \sum_{j=0}^{\infty}(\phi_j u - v_j)\right|^p \,dx\right)^{1/p} = \sum_{|\alpha|\le k}\left(\int_\Omega \left|\sum_{j=0}^{\infty} D^\alpha(\phi_j u - v_j)\right|^p \,dx\right)^{1/p} \\
&\le \sum_{|\alpha|\le k}\left(\int_\Omega \left(\sum_{j=0}^{\infty} \left|D^\alpha(\phi_j u - v_j)\right|\right)^p \,dx\right)^{1/p} \le \sum_{|\alpha|\le k}\sum_{j=0}^{\infty}\left(\int_\Omega \left|D^\alpha(\phi_j u - v_j)\right|^p \,dx\right)^{1/p} \\
&= \sum_{j=0}^{\infty}\sum_{|\alpha|\le k}\left(\int_\Omega \left|D^\alpha(\phi_j u - v_j)\right|^p \,dx\right)^{1/p} = \sum_{j=0}^{\infty}\|\phi_j u - v_j\|_{k,p} < \epsilon.
\end{aligned}$$

References

[GT83] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, New York, 1983.

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