SECOND DERIVATIVES OF THE NEWTONIAN POTENTIAL

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We assume $f : \Omega \to \mathbb{R}$ is locally Hölder continuous, that is, there exists $0 < \alpha \leq 1$ such that for each $K \subset \Omega$ compact there is a constant $C_K > 0$ such that

$$|f(x) - f(y)| \leq C_K\, |x - y|^{\alpha}, \quad \forall x, y \in K.$$

We also assume f is bounded in Ω.

The goal is to prove the representation formula (1) below for the second derivatives of the Newtonian potential.

Assume $n \geq 3$, let $\Gamma(x) = C_n |x|^{2-n}$ be the fundamental solution, $\eta : \mathbb{R} \to \mathbb{R}$ is smooth, $\eta(t) = 0$ for $t \leq 1$, $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $t \geq 2$ and $0 \leq \eta' \leq 2$.

Let Ω_0 be a domain for which the divergence theorem holds, and $\Omega \subset \Omega_0$, and let $\bar{f}(x) = f(x)$ for $x \in \Omega$ and $\bar{f}(x) = 0$ in $\Omega_0 \setminus \Omega$. Define for $x \in \Omega$

$$u(x) = \int_{\Omega_0} (D_{ij}\Gamma)(x - y)(\bar{f}(y) - f(x))\, dy - f(x) \int_{\partial\Omega_0} D_i\Gamma(x - y)\nu_j(y)\, d\sigma(y).$$

Since $|D_{ij}\Gamma(x)| \leq C_n |x|^{-n}$, f is bounded and locally Hölder continuous, the function $u(x)$ is well defined for all $x \in \Omega$. Let $\epsilon > 0$ and define

$$v_\epsilon(x) = \int_{\Omega} D_i\Gamma(x - y)\eta((x - y)/\epsilon) f(y)\, dy,$$

and let $v(x) = D_i w(x)$, where

$$w(x) = \int_{\Omega} \Gamma(x - y) f(y)\, dy.$$

If f is integrable in Ω, then by the homework, $v_\epsilon \in C^1(\mathbb{R}^n)$, and if in addition f is bounded, $w \in C^1(\mathbb{R}^n)$. We have

$$\begin{aligned}
D_j v_\epsilon(x) &= \int_\Omega D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right) f(y)\,dy \\
&= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right) \bar{f}(y)\,dy \\
&= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)(\bar{f}(y)-f(x))\,dy \\
&\qquad + f(x)\int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)\,dy \\
&= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)(\bar{f}(y)-f(x))\,dy \\
&\qquad - f(x)\int_{\Omega_0} D_{y_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)\,dy \\
&= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)(\bar{f}(y)-f(x))\,dy \\
&\qquad - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\eta((x-y)/\epsilon)\nu_j(y)\,d\sigma(y)
\end{aligned}$$

from the divergence theorem. Since $x \in \Omega$, if we take $\epsilon \le \operatorname{dist}(x,\partial\Omega_0)/2$, then $|x-y| \ge 2\epsilon$ for $y \in \partial\Omega_0$ and so $\eta((x-y)/\epsilon) = 1$. So

$$\begin{aligned}
D_j v_\epsilon(x) &= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)(\bar{f}(y)-f(x))\,dy \\
&\qquad - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j(y)\,d\sigma(y).
\end{aligned}$$

Then subtracting we get

$$\begin{aligned}
&u(x) - D_j v_\epsilon(x) \\
&= \int_{\Omega_0}\Big(D_{ij}\Gamma(x-y) - D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)\Big)(\bar{f}(y)-f(x))\,dy \\
&= \int_{\Omega_0} D_{x_j}\left[(1-\eta((x-y)/\epsilon))\,D_i\Gamma(x-y)\right](\bar{f}(y)-f(x))\,dy \\
&= \int_{|x-y|\le 2\epsilon} D_{x_j}\left[(1-\eta((x-y)/\epsilon))\,D_i\Gamma(x-y)\right](\bar{f}(y)-f(x))\,dy \\
&= \int_{|x-y|\le 2\epsilon}\left[D_{ij}\Gamma(x-y)\,(1-\eta((x-y)/\epsilon)) - D_i\Gamma(x-y)\frac{1}{\epsilon}\eta'(|x-y|/\epsilon)\frac{x_j-y_j}{x-y}\right](\bar{f}(y)-f(x))\,dy.
\end{aligned}$$

Let $K \subset \Omega$ be compact, and $K' = \{y : \operatorname{dist}(y, K) \leq \operatorname{dist}(K, \partial\Omega)/2\}$. We have K' is compact, $K' \subset \Omega$, and if $\epsilon < \operatorname{dist}(K, \partial\Omega)/2$, then $B_\epsilon(x) \subset K'$ for all $x \in K$. Therefore estimating the last integral we obtain for $x \in K$

$$\begin{aligned} |u(x) - D_j v_\epsilon(x)| &\leq \int_{|x-y|\leq 2\epsilon} \left[\left| D_{ij}\Gamma(x-y) \right| + \left| D_i\Gamma(x-y) \right| \frac{C}{\epsilon} \right] |f(y) - f(x)|\, dy \\ &\leq C_{K'} \int_{|x-y|\leq 2\epsilon} \left[\frac{C}{|x-y|^n} + \frac{C}{|x-y|^{n-1}} \frac{C}{\epsilon} \right] |x-y|^\alpha\, dy \leq C'_{K'} \epsilon^\alpha. \end{aligned}$$

Therefore $D_j v_\epsilon \to u$ uniformly on compact subsets of Ω as $\epsilon \to 0$, so u is continuous in Ω. In addition, $v_\epsilon \to v$ uniformly in $\mathbb{R}^n$ (this follows directly by subtracting). This implies that $w \in C^2(\Omega)$ and $u(x) = D_{ij}w(x)$ for $x \in \Omega$ by the fundamental theorem of calculus, because we write

$$v_\epsilon(x_1, \cdots, x_{j-1}, y_j, x_{j+1}, \cdots, x_n) = \int_{x_j}^{y_j} D_j v_\epsilon(x_1, \cdots, x_{j-1}, t, x_{j+1}, \cdots, x_n)\, dt + v_\epsilon(x),$$

and pass to the limit as $\epsilon \to 0$. So we have proved the following representation formula for the second derivatives of the Newtonian potential w for $x \in \Omega$:

$$D_{ij}w(x) = \int_{\Omega_0} (D_{ij}\Gamma)(x-y)(\bar{f}(y) - f(x))\, dy - f(x) \int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j(y)\, d\sigma(y), \tag{1}$$

under the assumption that f is bounded in Ω and locally Hölder continuous for some $0 < \alpha \leq 1$.

We next prove that $\Delta w = f$ in Ω. Let $x \in \Omega$, take R sufficiently large such that $\Omega \subset B_R(x)$, and apply (1) with $\Omega_0 = B_R(x)$, then

$$D_{ii}w(x) = \int_{|x-y|<R} (D_{ij}\Gamma)(x-y)(\bar{f}(y) - f(x))\, dy - f(x) \int_{|x-y|=R} D_i\Gamma(x-y)\nu_j(y)\, d\sigma(y),$$

adding over $1 \leq i \leq n$ and using that $\Delta\Gamma(x) = 0$ for $x \neq 0$, we obtain

$$\Delta w(x) = -f(x) \int_{|x-y|=R} D\Gamma(x-y) \cdot \nu(y)\, d\sigma(y).$$

We have $D\Gamma(x) = C_n(2-n)\dfrac{x}{|x|^n}$ and $\nu(y) = \dfrac{y-x}{|x-y|}$. Then $\Delta w(x) = f(x)C_n(2-n)\omega_n$, where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. Since $C_n = \dfrac{1}{(2-n)\omega_n}$, we are done.

We also have that $\Delta w(x) = 0$ for $x \notin \bar{\Omega}$, which follows differentiating twice under the integral sign (the justification follows from the homework).

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