MATH 8141, PDES
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PRELIMINARIES

NOTATION

Given $\alpha_1, ..., \alpha_n$ non-negative integers we set $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha! = \alpha_1!...\alpha_n!$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$. Given a complex-valued function $f(x)$, the support of f is the closure of the set $\{x : f(x) \neq 0\}$. Given f and g measurable, complex-valued functions defined over R^n the convolution of f and g is the function defined by

$$f * g(x) = \int_{R^n} f(x-y)g(y)\, dy.$$

Given Ω an open subset of R^n the following classes are defined:

$$C^N(\Omega)$$
$$= \{f : \Omega \to C; f \text{ and all its derivatives up the order } N \text{ are continuous functions in } \Omega\},$$
$$C^\infty(\Omega)$$
$$= \{f : \Omega \to C; f \text{ and all its derivatives of any order are continuous functions in } \Omega\},$$
$$C_0^\infty(\Omega) = \{f : \Omega \to C; f \in C^\infty(\Omega) \text{ and } \operatorname{supp}(f) \text{ is a compact subset of } \Omega\},$$
$$\mathcal{S} = \{f : R^n \to C; f \in C^\infty(R^n), \text{ and } \sup_{x \in R^n} |x^\alpha D^\beta f(x)| < \infty, \forall \alpha, \beta\}.$$

1. Taylor's Formula.

Let $\Omega \subset R^n$, Ω open and convex, $x_0 \in \Omega$, and let $f \in C^{N+1}(\Omega)$. Then for x in a neighborhood of x_0 we have

$$f(x) = \sum_{|\alpha| \le N} \frac{D^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha + R_N(x),$$

where

$$R_N(x) = \frac{1}{N!}\int_0^1 (1-t)^N \left(\frac{d}{dt}\right)^{N+1} \left(f(x_0 + t(x - x_0))\right)\, dt,$$

and

$$R_N(x) = o(|x - x_0|^{N+1}), \qquad \text{as } x \to x_0.$$

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2. Leibniz's Formula.

Let $P(x_1, ..., x_n)$ be a polynomial in n variables with complex coefficients. If $u(x)$ and $v(x)$ are C^∞-functions (or sufficiently smooth) we have

$$P(D)(uv)(x) = \sum_\alpha \frac{P^{(\alpha)}(D)u(x)}{\alpha!} D^\alpha v(x).$$

In particular

$$D^\beta(uv)(x) = \sum_\alpha \frac{\beta!}{\alpha!(\beta-\alpha)!} D^{\beta-\alpha}u(x) D^\alpha v(x).$$

To prove this formula we may note the fact that

$$P(D)e^{x\cdot\xi} = P(\xi)e^{x\cdot\xi},$$

where $x \cdot \xi = \sum_{i=1}^n x_i \xi_i$.

3. The Inverse Function Theorem.

Let $\Omega_1 \subset R^n$, $\Omega_2 \subset R^m$, open sets, and $u : \Omega_1 \to \Omega_2$. We say that the function u is a C^k-diffeomorphism if u is bijective, $u \in C^k(\Omega_1)$, and its inverse function $u^{-1} \in C^k(\Omega_2)$.

Let $\Omega \subset R^n$ open, $a \in \Omega$, and $f : \Omega \to R^n$ is function such that $f \in C^k(\Omega)$, $f(x) = (f_1(x), ..., f_n(x))$. If the Jacobian

$$J_f(a) = \det\left(\frac{\partial f_i}{\partial x_j}(a)\right) \neq 0,$$

then there exist a neighborhood U of a and a neighborhood V of $b = f(a)$ such that $f/U : U \to V$ is a C^k-diffeomorphism.

4. Implicit Function Theorem.

Let $\Omega \subset R^n \times R^m$ an open set, and let $f : \Omega \to R^m$ be a C^k-function in Ω, $f(x, y) = (f_1(x, y), ..., f_m(x, y))$, $x \in R^n$ and $y \in R^m$. Let $(a, b) \in \Omega$ such that

$$f(a, b) = 0,$$

and suppose that the Jacobian

$$J_y f(a, b) = \det\left(\frac{\partial f_i}{\partial y_j}(a, b)\right) \neq 0.$$

Then there exist a neighborhood U of a, a neighborhood V of b, and a function $g : U \to V$, $g \in C^k(U)$ such that

$$g(a) = b,$$

and

$$f(x, g(x)) = 0, \qquad \forall x \in U.$$

5. The Divergence Theorem.

Let $\Omega \subset R^n$ be a bounded, open and connected set such that its boundary $\partial\Omega$ is C^1. Let ν denote the unit outward normal to $\partial\Omega$. Given any vector field $W(x) = (w_1(x), ..., w_n(x))$ which is $C^1(\overline{\Omega})$, we then have

$$\int_\Omega \operatorname{div} W \, dx = \int_{\partial\Omega} W \cdot \nu \, d\sigma(x).$$

6. Differentiation under the integral sign.

Let $\Omega \subset R^n$ measurable, and $x_0 \in R^n$. Suppose that $f(x,t)$, $x, t \in R^n$ is a function such that

$$f(x, \cdot) \in L^1(\Omega), \qquad \text{for } |x - x_0| < \epsilon,$$

and the partial derivative $\dfrac{\partial f}{\partial x_j}(x,t)$ exists and there exists $g(t) \in L^1(\Omega)$ such that

$$\left| \frac{\partial f}{\partial x_j}(x,t) \right| \leq g(t),$$

for $t \in \Omega$ and $|x - x_0| < \epsilon$.

Then the function

$$F(x) = \int_\Omega f(x,t) \, dt$$

is differentiable with respect to x_j in Ω and

$$\frac{\partial F}{\partial x_j}(x) = \int_\Omega \frac{\partial f}{\partial x_j}(x,t) \, dt.$$

7. Minkowsky's inequality.

Let $1 \leq p < \infty$. If $f(x,y)$ is a measurable function in $R^n \times R^m$ then

$$\left(\int_{R^n} \left(\int_{R^m} |f(x,y)| \, dx \right)^p dy \right)^{1/p} \leq \int_{R^m} \left(\int_{R^n} |f(x,y)|^p \, dy \right)^{1/p} dx.$$

8. Hölder's inequality.

Let $1 \leq p \leq \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then

$$\int_{R^n} |f(x)g(x)| \, dx \leq \|f\|_p \|g\|_q.$$

9. Young's inequality.
Let $1 \leq p \leq \infty$, $f \in L^1(R^n)$ and $g \in L^p(R^n)$. Then

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

10. Let $1 \leq p \leq \infty$, $f \in L^p(R^n)$ and $g \in L^q(R^n)$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then

$$\|f * g\|_\infty \leq \|f\|_p \|g\|_q.$$

11. Let f be a function defined over R^n, $h \in R^n$, and the translation operator $\tau_h f(x) = f(x - h)$. If $1 \leq p < \infty$ and $f \in L^p(R^n)$ then

$$\|\tau_h f - f\|_p \to 0, \qquad \text{as } h \to 0.$$

If $p = \infty$ the result is false.

12. Let $\phi \in L^1(R^n)$, $\int_{R^n} \phi(x)\, dx = a$, $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. If $1 \leq p < \infty$, and $f \in L^p(R^n)$ then

$$f * \phi_\epsilon \to af, \qquad \text{in } L^p\text{-norm.}$$

If $f \in L^\infty$ and f is uniformly continuous on $A \subset R^n$ then

$$f * \phi_\epsilon \to af, \qquad \text{uniformly in } A.$$

If $a = 1$ then the family ϕ_ϵ is called an approximation of the identity.

13. Let $1 \leq p \leq \infty$. If $f \in L^p$ and $\phi \in \mathcal{S}$ then $f * \phi \in C^\infty$ and we have

$$D^\alpha(f * \phi)(x) = f * D^\alpha \phi(x), \qquad \forall \alpha.$$

14. Let

$$f(t) = \begin{cases} e^{1/(1-t^2)}, & \text{for } |t| < 1 \\ 0, & \text{for } |t| \geq 1. \end{cases}$$

Then $f \in C_0^\infty(R^n)$. Show that there exists $f \in C_0^\infty(R^n)$ such that

$$\operatorname{supp}(f) = B_r(x_0) = \{x \in R^n : |x - x_0| \leq r\}.$$

15. $\operatorname{supp}(f * g) \subset \operatorname{supp}(f) + \operatorname{supp}(g) = \{x + y : x \in \operatorname{supp}(f),\ y \in \operatorname{supp}(g)\}$.

16. The class $C_0^\infty(R^n)$ is dense in $L^p(R^n)$, $1 \leq p < \infty$.