**PDES, Math 8141 (old 561), Prof. Gutiérrez** **Homework 4 (Due 11/5/09)**

1. (*Differentiation under the integral sign*) Let $k(x,y)$ be a function defined for all $x \in \mathcal{O} \subset \mathbb{R}^m$ open and for a.e. $y \in \mathbb{R}^n$ such that $k(x,\cdot) \in L^1(\mathbb{R}^n)$ for all $x \in \mathcal{O}$. Prove that

   (a) The function
   $$u(x) = \int_{\mathbb{R}^n} k(x,y)\,dy$$
   is well defined for all $x \in \mathcal{O}$.

   (b) Fix $1 \le i \le m$. Assume that $\partial_{x_i} k(x,y)$ exists for all $x \in \mathcal{O}$ and for almost all $y \in \mathbb{R}^n$, and there exists $g \in L^1(\mathbb{R}^n)$ such that
   $$|\partial_{x_i} k(x,y)| \le g(y), \qquad \forall x \in \mathcal{O}, \qquad \text{and for almost all } y \in \mathbb{R}^n.$$
   Assume further that
   $$\partial_{x_i} k(\cdot,y) \in C(\mathcal{O}) \quad \text{and for almost all } y \in \mathbb{R}^n.$$
   Then the derivative $\partial_{x_i} u$ exists for all $x \in \mathcal{O}$ and
   $$\partial_{x_i} u(x) = \int_{\mathbb{R}^n} \partial_{x_i} k(x,y)\,dy,$$
   for all $x \in \mathcal{O}$.

   (c) If the gradient $D_x k(\cdot,y) \in C(\mathcal{O})$ and there exists $g \in L^1(\mathbb{R}^n)$ such that
   $$|D_x k(x,y)| \le g(y), \qquad \forall x \in \mathcal{O}, \qquad \text{and for almost all } y \in \mathbb{R}^n,$$
   then
   $$D_x u(x) = \int_{\mathbb{R}^n} D_x k(x,y)\,dy,$$
   for all $x \in \mathcal{O}$ and $u \in C(\mathcal{O})$.

   (d) If $\partial_{x_j}\partial_{x_i} k(\cdot,y) \in C(\mathcal{O})$ for a.e. $y \in \mathbb{R}^n$, and there exists $g \in L^1(\mathbb{R}^n)$ such that
   $$|\partial_{x_j}\partial_{x_i} k(x,y)| \le g(y), \qquad \forall x \in \mathcal{O}, \qquad \text{and for almost all } y \in \mathbb{R}^n,$$
   then
   $$\partial_{x_j}\partial_{x_i} u(x) = \int_{\mathbb{R}^n} \partial_{x_j}\partial_{x_i} k(x,y)\,dy.$$

2. (*The Newtonian potential is $C^1$*) Let $\eta(t)$ be a smooth function such that $\eta(t) = 0$ for $t \le 1$, $0 \le \eta \le 1$ everywhere, and $\eta(t) = 1$ for $t \ge 2$. Let $n \ge 3$, $\Gamma(x) = \dfrac{1}{|x|^{n-2}}$ (for $n = 2$, $\Gamma(x) = \log|x|$) and consider the function
   $$\Gamma_\epsilon(x) = \Gamma(x)\,\eta(|x|/\epsilon),$$
   where $\epsilon > 0$.

(a) If $\rho \in L^1(\mathbb{R}^n)$, then prove that the function

$$w_\epsilon(x) = \int_{\mathbb{R}^n} \rho(y)\Gamma_\epsilon(x-y)\,dy$$

is well defined for all $x \in \mathbb{R}^n$.

(b) If $\rho \in L^1(\mathbb{R}^n)$, then prove that $w_\epsilon \in C^1(\mathbb{R}^n)$ (apply problem 1 with $k(x,y) = \rho(y)\Gamma_\epsilon(x-y)$).

(c) If $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, prove that

$$w_\epsilon(x) \to w(x) = \int_{\mathbb{R}^n} \rho(y)\Gamma(x-y)\,dy$$

uniformly for $x \in \mathbb{R}^n$, where $w(x)$ is well defined for all $x$.

(d) If $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then prove that

$$Dw_\epsilon \to v$$

uniformly in $\mathbb{R}^n$, where

$$v(x) = \int_{\mathbb{R}^n} \rho(y)D_x(\Gamma(x-y))\,dy = (v_1(x), \cdots, v_n(x)),$$

where the last integral is well defined.

(e) As a consequence of (b), (c) and (d), the functions $w$ and $v$ are continous when $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

(e) If $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then $w \in C^1(\mathbb{R}^n)$, and $Dw(x) = v(x)$.
HINT: by the fundamental theorem of calculus, $w_\epsilon(x_1, x_2, \cdots, x_n) = \int_{y_1}^{x_1} \partial_{x_1} w_\epsilon(t, x_2, \cdots, x_n)\,dt + w_\epsilon(y_1, x_2, \cdots, x_n)$, and from the uniform convergence $w(x_1, x_2, \cdots, x_n) = \int_{y_1}^{x_1} v_1(t, x_2, \cdots, x_n)\,dt + w(y_1, x_2, \cdots, x_n)$.