PDES, Math 8141 (old 561), Prof. Gutiérrez **Homework 3 (Due 10/22/09)**

1. Let C be a cone in $\mathbb{R}^n$ whose base is a set Ω that lies on a hyperplane. Suppose that the height of the cone is h. Use the vector field $F(x) = x$ and the divergence theorem to show that C has volume
$$\operatorname{vol}(C) = \frac{\operatorname{area}(\Omega)\, h}{n}.$$

2. Consider in $\mathbb{R}^n$, $n \geq 2$, the vector field $F(x) = f(|x|)\dfrac{x}{|x|}$ where $f(t)$ is a continuous function for $t > 0$. Assume that $\operatorname{div}F(x) = 0$ for all $x \neq 0$. Show that $f(t) = \dfrac{f(1)}{t^{n-1}}$. In particular, this shows that in 3-d, the Newtonian vector field is the only central field of force with divergence 0.

3. Let F be a vector field, $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(x) = (F_1(x), \ldots, F_n(x))$. Assume that F is C^1. The purpose of this problem is to see how the divergence behaves under changes of coordinates in $\mathbb{R}^n$. Consider the matrix
$$F'(x) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1}(x) & \cdots & \dfrac{\partial F_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_n}{\partial x_1}(x) & \cdots & \dfrac{\partial F_n}{\partial x_n}(x) \end{pmatrix},$$
and note that
$$\operatorname{div}F(x) = \operatorname{trace}\left(F'(x)\right).$$
Let $A = \left(a_{ij}\right) \in \mathbb{R}^{n\times n}$ and for $z = (z_1, \ldots, z_n)$ we set $Az = \left(\sum_{j=1}^n a_{1j}z_j, \ldots, \sum_{j=1}^n a_{nj}z_j\right)$.

   (a) Let $H(z) = F(Az)$. Show that
   $$H'(z) = \left(F'(Az)\right) A.$$

   (b) Let $B \in \mathbb{R}^{n\times n}$ and $G(z) = BF(z)$. Show that
   $$G'(z) = BF'(z).$$

   (c) Let $K(x) = BF(Ax)$. Prove that
   $$K'(x) = BF'(Ax)A$$
   and
   $$\operatorname{div}K(x) = \operatorname{trace}\left(AB\left(F'(Ax)\right)\right),$$
   Thus, if $AB = Id$ then
   $$\operatorname{div}K(x) = \operatorname{div}F(Ax).$$

   (d) Let $E = \{e_1, \ldots, e_n\}$ and $V = \{v_1, \ldots, v_n\}$ be two orthonormal basis of $\mathbb{R}^n$. Then if a point P has coordinates $(x_1, \ldots, x_n)$ in the base E then the coordinates of P in the basis V are $(\xi_1, \ldots, \xi_n) = O(x_1, \ldots, x_n)$ where O is an orthogonal matrix. Assume that $F : (\mathbb{R}^n, V) \to (\mathbb{R}^n, V)$, that means if P is a point in $\mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$ in the base V then $F(P)$ is a point with coordinates $F(x_1, \ldots, x_n)$ in the base V. If we change the basis, we have $O : (\mathbb{R}^n, E) \to (\mathbb{R}^n, V)$ and $O^t : (\mathbb{R}^n, V) \to (\mathbb{R}^n, E)$. Then if a point P has coordinates $(\xi_1, \ldots, \xi_n)$ in the basis E then the coordinates of $F(P)$ in the basis E are $K(\xi) = O^t F(O\xi)$. Then (c) shows that the divergence is a quantity invariant by orthogonal changes of coordinates.