PDES, Math 8141 (old 561), Prof. Gutiérrez   Homework 2 (Due 10/8/09)

Newton's law of gravity: it m_1 and m are masses located at the points P_1 and P respectively, the force felt at P is given by

$$F(P) = G\, m_1\, m\, \frac{P_1 - P}{|P_1 - P|^3},$$

where $G = 6.674\times 10^{-11}\, N(m/kg)^2$ is the universal constant of gravitation, see `http://en.wikipedia.org/wiki/Gravitational_constant`.

Coulomb's law: it q_1 and q are charges located at the points P_1 and P respectively, the force felt at P is given by

$$F(P) = \frac{1}{4\pi\epsilon_0}\, q_1\, q\, \frac{P_1 - P}{|P_1 - P|^3},$$

where $\dfrac{1}{4\pi\epsilon_0} = 8.854 \times 10^9\, N\,(m/coulomb)^2$, see `http://en.wikipedia.org/wiki/Coulomb's_law`.

1. The gravitational vector field created by a wire. Let $\gamma(t)$, $a \le t \le b$ be a C^1 curve in $\mathbb{R}^3$. Suppose we have a wire with the shape of γ and with homogeneous density λ. That is, the mass of an arc of length α is $\lambda\alpha$. Show that the gravitational vector field due to the wire is

$$F(x) = \lambda\, G \int_a^b |\gamma'(t)|\, \frac{\gamma(t) - x}{|\gamma(t) - x|^3}\, dt.$$

2. The electrostatic vector field due to a hollow sphere with center 0 and radius R, $S(0,R)$, that is uniformly charged with density λ is given by

$$F(x) = \frac{\lambda}{4\pi\epsilon_0} \int_{|z|=R} \frac{x - y}{|x - y|^3}\, d\sigma(y).$$

Prove that $F = 0$ in the interior of $S(0,R)$.

3. Consider in $\mathbb{R}^3$ the ball $B(0,R)$. Assume that the ball is homogeneous with density λ. The Newtonian potential due to this ball is given by

$$U(x) = G \int_{B(0,R)} \lambda\, \frac{1}{|y - x|}\, dy.$$

   (a) Show that

$$U(x) = \begin{cases} G\lambda \dfrac{4}{3}\pi R^3 \dfrac{1}{|x|}, & \text{for } |x| > R \\ G\lambda \dfrac{2}{3}\pi (3R^2 - |x|^2), & \text{for } 0 \le |x| \le R. \end{cases}$$

   (b) Prove that the first order derivatives of U exist and are continuous everywhere.

   (c) Prove that all the second derivatives of U exist and are continuous in $\mathbb{R}^3 \setminus \{x : |x| = R\}$. Prove that they do not exist on $|x| = R$.

(d) Prove that

$$\Delta U(x) = \operatorname{div} DU(x) = \begin{cases} -G\lambda 4\pi, & \text{for } 0 \le |x| < R \\ 0, & \text{for } |x| > R. \end{cases}$$

4. Let $R > 0$ and $P = (R/4, 0, 0)$. Consider the solid $\Omega = B(0, R) \setminus B(P, R/4)$ and assume that it is homogenous with density λ. Let U be the Newtonian potential due to Ω. Use problem 3 to calculate U in $B(P, r/4)$. Conclude that the Newtonian vector field due to Ω is constant in $B(P, R/4)$.

   Answer: $U(x, y, z) = G\lambda \frac{R}{3}\pi \left(\frac{23}{4}R - x\right), (x, y, z) \in B(P, R/4)$.