PDES I, Math 8141 (old 561)
Prof. Gutiérrez
Homework 1, First order pdes (Due on 9/22/09)

1. Suppose $\left|\frac{\partial F}{\partial x}(t,x)\right| \le h(t)$, and $x(t)$ and $y(t)$ are two solutions defined in a neighborhood of $t = 0$ of $x' = F(t,x)$. Prove that

$$|x(t) - y(t)| \le |x(0) - y(0)| \exp\left(\int_0^t h(s)\, ds\right).$$

2. Let $F(x)$ be $C^1(\mathbb{R}^n)$. Prove that any solution to $x' = F(x)$ is C^2. Prove this is might not be true when F also depends on t.

3. Solve the following Cauchy problems:

   1. $u_x + (x+y)u_y = 1, u(x,-x) = 0$; ANS: $u = \ln(x+y+1)$;
   2. $(1+x^2)u_x + 2xyu_y = 0, u(x, x+x^3) = h(x)$; ANS: $u = h(y/(1+x^2))$;
   3. $(y+1)u_x + (x+1)u_y = u^2$ passing through the curve $(s, -s, 1/\log s)$, $s > 0$; ANS: $u = 1/(\ln((x-y)/2))$;
   4. $x^2u_x - y^2u_y + 2(x-y)u = 0, u(x,x) = x$; ANS: $u = 8(xy)^2/((x+y)^3)$;
   5. $u_x = \left(u_y\right)^2, u(0,y) = y^2/2$; ANS: $u = y^2/(2(1-2x))$;
   6. $xu_x + yu_y + u_xu_y = u, u(s,0) = s^2$; ANS: $u = (4x-y)^2/16$;
   7. $x\,(u_x)^2 + yu_y = 0, u(s,1) = -s$; ANS: $u = x/((\ln y) - 1)$;
   8. $x\,(u_x)^2 + \left(u_y\right)^3 = 1, u(s,0) = \sqrt{s}, s > 0$; ANS: $u = \sqrt{x} + (3/4)^{1/3}y$.