Real Analysis II, Prof. Gutiérrez **Test 2, Due May 12, 11:30AM, 2009**

Each problem is worth 10 points. Justify all your answers.

1. Let (X, Σ, μ) be a measure space with $\mu(X) = 1$, and $f, g \in L^1(X, \mu)$ nonnegative satisfying $f(x)g(x) \geq 1$ for a.e. $x \in X$. Prove that
$$\int_X f(x)\, d\mu(x) \int_X g(x)\, d\mu(x) \geq 1.$$

2. Let μ, ν, ω be measures defined in a common σ-algebra Σ of subsets of X. Prove that
   1. If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu \equiv 0$.
   2. If $\nu \ll \mu$ and $\mu \ll \omega$, then $\nu \ll \omega$.
   3. If $\mu \perp \omega$ and $\nu \ll \mu$, then $\nu \perp \omega$.

3. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of distinct points in $\mathbb{R}^n$ and $\{a_j\}_{j=1}^{\infty}$ is a numerical sequence such that $\sum_{j=1}^{\infty} |a_j| < \infty$. Prove that
$$\Phi = \sum_{j=1}^{\infty} a_j\, \delta_{x_j}$$
is an additive set function on the σ-algebra of all subsets of $\mathbb{R}^n$ that is singular with respect to Lebesgue measure. Here δ_{x_j} denotes the Dirac delta measure concentrated at x_j, that is, $\delta_{x_j}(A) = 1$ if $x_j \in A$ and is zero otherwise.

4. Let μ be a Borel measure on $(0, +\infty)$ that is absolutely continuous with respect to the one dimensional Lebesgue measure dx. Suppose that for each Borel set $A \subset (0, +\infty)$ and for each $\alpha > 0$ we have $\mu(\alpha A) = \mu(A)$, where $\alpha A = \{\alpha x : x \in A\}$. If the Radon-Nikodym derivative $\dfrac{d\mu}{dx}$ is a continuous function, then prove that $\dfrac{d\mu}{dx} = \dfrac{C}{x}$ for all $x > 0$ and some constant $C \geq 0$.

5. Let μ^* be an outer measure on X. Prove that $E \subset X$ is Carathéodory measurable if and only if for each $\epsilon > 0$ there exists a Carathéodory measurable set $F \subset E$ such that $\mu^*(E \setminus F) < \epsilon$.

6. Let μ be a translation invariant Borel measure in $\mathbb{R}^n$, that is, $\mu(E + x) = \mu(E)$ for each Borel set E and each $x \in \mathbb{R}^n$. Suppose further that μ is finite on compact sets. Prove that μ is a multiple of the Lebesgue measure.

   HINT:

   1. If Q_a denotes a translate of the cube $\{x : 0 \leq x_j \leq a, j = 1, \cdots, n\}$ and $\mu(Q_1) = c$, then prove that $\mu(Q_{1/k}) = c k^{-n}$ for each integer k. (Prove that Q_1 can be written as a disjoint union of k^n translates of $Q_{1/k}$.)

2. Show that μ is absolutely continuous w.r.t. Lebesgue measure, and by Radon-Nikodym there exists $f \in L^1_{loc}(\mathbb{R}^n)$ such that $\mu(E) = \int_E f(x)\,dx$.

3. By Lebesgue differentiation theorem f is constant a.e.

7. Let $\alpha > 0$ and H_α be the Hausdorff outer measure in $\mathbb{R}^n$. Given $A \subset \mathbb{R}^n$ and $t > 0$, let $\delta_t A = \{tx : x \in A\}$. Prove that $H_\alpha(\delta_t A) = t^\alpha\, H_\alpha(A)$.

8. Prove that

   1. $H_\alpha(\mathbb{R}^n) = +\infty$ for all $0 < \alpha \leq n$.

   2. if $\alpha \leq \beta$, then $H_\alpha(A) \geq H_\beta(A)$ for all $A \subset \mathbb{R}^n$.

9. Prove that the Hausdorff measure is not σ-finite for $0 < \alpha < n$.

10. Prove that the Hausdorff dimension of the set A equals $\dim_{\mathbb{H}}(A) = \inf\{\alpha : H_\alpha(A) < +\infty\} = \sup\{\alpha : H_\alpha(A) = +\infty\}$, and $\dim_{\mathbb{H}}\left(\cup_{j=1}^\infty A_j\right) = \sup_j \dim_{\mathbb{H}}(A_j)$.