Real Analysis Problems

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1 Continuity

Problem 1.1 Let r_n be the sequence of rational numbers and

$$f(x) = \sum_{\{n : r_n < x\}} \frac{1}{2^n}.$$

Prove that

1. f is continuous on the irrationals.
2. f is discontinuous on the rationals.
3. Calculate $\int_0^1 f(x)\,dx$.

Hint: for (3) set $A(x) = \{n : r_n < x\}$ so $f(x) = \sum_{n=1}^{\infty} \chi_{A(x)}(n) \dfrac{1}{2^n}$; use Fubini.

Problem 1.2 Let $f_n(x) = \sin\sqrt{x + 4\,n^2\,\pi^2}$ on $[0, +\infty)$. Prove that

1. f_n is equicontinuous on $[0, +\infty)$.
2. f_n is uniformly bounded.
3. $f_n \to 0$ pointwise on $[0, +\infty)$.
4. There is no subsequence of f_n that converges to 0 uniformly.
5. Compare with Arzelà-Ascoli.

Problem 1.3 Prove that the class of Lipschitz functions f in $[a, b]$ with Lipschitz constant $\leq K$ and $f(a) = 0$ is a compact set in $C([a, b])$.

Problem 1.4 Let B be the unit ball in $C([a, b])$. Define for $f \in C([a, b])$

$$Tf(x) = \int_a^b \left(-x^2 + e^{-x^2+y}\right) f(y)\,dy.$$

Prove that $T(B)$ is relatively compact in $C[a, b]$.

Problem 1.5 Let $g_n : \mathbb{R} \to \mathbb{R}$ be the continuous function that is zero outside the interval $[0, 2/n]$, $g_n(1/n) = 1$, and g_n is linear on $(0, 1/n)$ and $(1/n, 2/n)$. Prove that $g_n \to 0$ pointwise in $\mathbb{R}$ but the convergence is not uniform on any interval containing 0. Let r_k be the rational numbers and define

$$f_n(t) = \sum_{k=1}^{\infty} 2^{-k}\, g_n(t - r_k).$$

Prove that

1. f_n is continuous on $\mathbb{R}$.
2. $f_n \to 0$ pointwise in $\mathbb{R}$.
3. f_n does not converge uniformly on any interval of $\mathbb{R}$.

Problem 1.6 Let $f \in C(\mathbb{R})$ and $f_n(x) = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right)$. Show that f_n converges uniformly on every finite interval.

Problem 1.7 Consider the following statements:

(a) f is a continuous function a.e. on $[0, 1]$

(b) there exists g continuous on $[0, 1]$ such that $g = f$ a.e.

Show that (a) does not imply (b), and (b) does not imply (a).

Problem 1.8 Show that if f and g are absolutely continuous functions in $[a, b]$ and $f'(x) = g'(x)$ a.e., then $f(x) - g(x) = \text{constant}$, for each $x \in [a, b]$.

Problem 1.9 If f is absolutely continuous in $[0, 1]$, then f^2 is absolutely continuous in $[0, 1]$.

Problem 1.10 Let $0 < \alpha \leq 1$. A function $f \in C^\alpha([0, 1])$, f *is Hölder continuous of order* α, if there exists $K \geq 0$ such that $|f(x) - f(y)| \leq K\,|x - y|^\alpha$ for all $x, y \in [0, 1]$. Prove that

1. $C^\alpha[0, 1] \subset C^\beta([0, 1])$ for $0 < \beta \leq \alpha \leq 1$.
2. If f is Hölder continuous of order one, that is, f is Lipschitz, then f is absolutely continuous on $[0, 1]$.

3. There exists $f \in C^\alpha([0,1])$ for all $0 < \alpha < 1$ such that f is not absolutely continuous on $[0,1]$. Take for example $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

Problem 1.11 Consider the space $BV[a,b]$ of bounded variation functions in $[a,b]$ with the norm

$$\|f\|_{BV} = \sup_{\Gamma} \sum_{i=1}^{k} |f(\alpha_i) - f(\alpha_{i-1})| + |f(a)|,$$

where the supremum is taken over all partitions $\Gamma = \{\alpha_0, \dots, \alpha_k\}$ of $[a,b]$. Prove that

1. $(BV[a,b], \|\cdot\|)$ is a Banach space;
2. the set of absolutely continuous functions in $[a,b]$ is a closed subspace of $BV[a,b]$.

Problem 1.12 Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|}$$

converges for each $x \in \mathbb{R}$ and the sum is a Lipschitz function.

Problem 1.13 Let $f : [a,b] \to \mathbb{R}$ be continuous and $\{x_k\} \subset [a,b]$ a Cauchy sequence. Prove that $f(x_k)$ is a Cauchy sequence.

Problem 1.14 Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq M\,|x-y|^\alpha$ for some $M > 0$, $\alpha > 1$ and for all $x, y \in \mathbb{R}$. Prove that f is constant.

Problem 1.15 Let $f(x) = x^2 \sin(1/x^2)$ for $x \in [-1,1]$, $x \neq 0$, and $f(0) = 0$. Show that f is differentiable on $[-1,1]$ but f' is unbounded on $[-1,1]$.

Problem 1.16 Suppose $f_n \to f$ uniformly in Ω open, and $\{x_n\} \subset \Omega$ with $x_n \to x \in \Omega$. Prove that $f_n(x_n) \to f(x)$.

Problem 1.17 If x_n is a sequence such that $|x_{n+1} - x_n| \leq 1/2^n$, then $\{x_n\}$ is a Cauchy sequence.

Problem 1.18 Let $a < b, c < d$. Define

$$f(x) = \begin{cases} ax\sin^2\frac{1}{x} + bx\cos^2\frac{1}{x}, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ cx\sin^2\frac{1}{x} + dx\cos^2\frac{1}{x}, & \text{for } x > 0. \end{cases}$$

Calculate D^-f, D^+f, D_-f, D_+f at $x = 0$.

Problem 1.19 Let f be a continuous function in [-1,2]. Given $0 \le x \le 1$, and $n \ge 1$ define the sequence of functions

$$f_n(x) = \frac{n}{2}\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\,dt.$$

Show that f_n is continuous in [0,1] and f_n converges uniformly to f in [0,1].

Problem 1.20 Let $f_n : \mathbb{R} \to \mathbb{R}$ be a uniformly bounded sequence of functions. Show that for each countable subset $S \subset R$ there exists a subsequence of f_n which converges in S.

Hint: select the subsequence by using a diagonal process

Problem 1.21 Let f be a continuous function in $[0,1]$ such that f is absolutely continuous in $[0,\epsilon]$ for every $\epsilon, 0 < \epsilon < 1$. Show that f is absolutely continuous in $[0,1]$.

Problem 1.22 Let f_n be absolutely continuous functions in $[a,b]$, $f_n(a) = 0$. Suppose f_n' is a Cauchy sequence in $L^1([a,b])$. Show that there exists f absolutely continuous in $[a,b]$ such that $f_n \to f$ uniformly in $[a,b]$.

Problem 1.23 Let $f_n(x) = \cos(n\,x)$ on $\mathbb{R}$. Prove that there is no subsequence f_{n_k} converging uniformly in $\mathbb{R}$.

2 Semicontinuous functions

Definition 2.1 *Let $\Omega \subset \mathbb{R}^n$ be open. The function $f : \Omega \to \overline{\mathbb{R}}$ is lower (upper) semicontinuous if $f(z) \leq \liminf_{x\to z} f(x) = \lim_{\delta\to 0} \inf_{|x-z|<\delta} f(x)$ ($f(z) \geq \limsup_{x\to z} f(x) = \lim_{\delta\to 0} \sup_{|x-z|<\delta} f(x)$).*

Problem 2.2 Prove that the following statements are equivalent:

1. f is lower semicontinuous;
2. $f^{-1}(c, +\infty)$ is open for each $c \in \mathbb{R}$;
3. $f^{-1}(-\infty, c]$ is closed for each $c \in \mathbb{R}$;
4. If $x_k \to z$, then $f(z) \leq \sup_k f(x_k)$;
5. For each $z \in \Omega$ and for each $\epsilon > 0$ there exists a neighborhood V of z such that $f(x) \geq f(z) - \epsilon$ for each $x \in V$.

Problem 2.3 Let $g : \Omega \to \mathbb{R}$ and define

$$f(x) = \liminf_{z\to x} g(z).$$

Prove that f is lower semicontinuous.

Problem 2.4 Prove that

1. if $E \subset \mathbb{R}^n$, then χ_E is lower semicontinuous if and only if E is open.
2. the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \dfrac{1}{n} & \text{if } x = m/n \text{ with } n \in \mathbb{N},\ m \in \mathbb{Z}, \text{ and } m/n \text{ irreducible} \end{cases}$$

   is upper semicontinuous.

Problem 2.5 Let $\{f_\alpha\}_{\alpha\in J}$ be a family of lower semicontinuous function in Ω. Prove that $f(x) = \sup_{\alpha\in J} f_\alpha(x)$ is lower semicontinuous.

Problem 2.6 Prove that the function defined in Problem 1.1 is lower semicontinuous in $\mathbb{R}$.

3 Measure

Problem 3.1 Let E_j be a sequence of sets in $\mathbb{R}^n$, $E^* = \limsup_{j\to\infty} E_j = \bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty} E_j$, and $E_* = \liminf_{j\to\infty} E_j = \bigcup_{k=1}^{\infty}\bigcap_{j=k}^{\infty} E_j$. Show that $\overline{\lim}_{j\to\infty}\chi_{E_j}(x) = \chi_{E^*}(x)$, and $\underline{\lim}_{j\to\infty}\chi_{E_j}(x) = \chi_{E_*}(x)$.

Prove that if E_j are measurable and $|E_j| \to 0$, then $\underline{\lim}_{j\to\infty}\chi_{E_j}(x) = 0$ a.e. Notice that this does not imply that $\overline{\lim}_{j\to\infty}\chi_{E_j}(x) = 0$ a.e. If for example, in $[0,1]$ we let $E_1^1 = [0,1/2]$, $E_2^1 = [1/2,1]$, $E_1^2 = [0,1/4], E_2^2 = [1/4,1/2], E_3^2 = [1/2,3/4], E_4^2 = [3/4,1]$, and so on, then the characteristics functions of these sets have upper limit one for each $x \in [0,1]$. However, if $\sum_{k=1}^{\infty}|E_j| < \infty$, then $\overline{\lim}_{j\to\infty}\chi_{E_j}(x) = 0$ a.e. (Borel-Cantelli).

Problem 3.2 Consider the set of rational numbers $\mathbb{Q} = \{q_j\}_{j=1}^{\infty}$. Prove that

$$\mathbb{R} \setminus \bigcup_{j=1}^{\infty}\left(q_j - \frac{1}{j^2}, q_j + \frac{1}{j^2}\right) \neq \emptyset.$$

Problem 3.3 Construct a countable family of closed intervals contained in $[0,1]$ such that the union covers $[0,1]$ but there is no finite subcovering.

Problem 3.4 Let E be a set in $\mathbb{R}^n$. Show that there exists a sequence G_i of open sets, $G_1 \supset G_2 \supset \cdots \supset E$ such that

$$|\bigcap_{i=1}^{\infty} G_i| = |E|_e.$$

Problem 3.5 If $E \subset [0,1]$, $|E| = 0$ and $f(x) = x^3$, then $|f(E)| = 0$.

Problem 3.6 True or false, justify your answer.

1. The class of Lebesgue measurable sets has cardinality 2^c.
2. Every perfect set has positive measure.
3. Every bounded function is measurable.

Problem 3.7 Let μ^* be an exterior measure and let A_n be a sequence of sets such that

$$\sum_{n=1}^{\infty} \mu^*(A_n) < +\infty.$$

Show that $\mu^*(\overline{\lim} A_n) = 0$. ($\overline{\lim} A_n = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j$)

Problem 3.8 Let $f : [0,1] \to [0,1]$ be the Cantor function, and let μ be the Lebesgue-Stieltjes measure generated by f, i.e. $\mu((a,b]) = f(b) - f(a)$, for $(a,b] \subset [0,1]$. Show that μ is singular with respect to Lebesgue measure.

Problem 3.9 Let Ω be an uncountable set. The class of sets defined by

$$F = \{A \subset \Omega : A \text{ is countable or } A^c \text{ is countable}\}$$

is a σ-algebra. Define the measures: $\mu(A) = +\infty$ if A is infinite, $\mu(A) = \#(A)$ if A is finite; $\nu(A) = 0$ if A is countable and $\nu(A) = 1$ if A is uncountable. Show that $\nu \ll \mu$ but an integral representation of the form $\nu(A) = \int_A f \, d\mu$ is not valid. Why this does not contradict the Radon-Nikodym theorem?

Problem 3.10 Show that given $\delta, 0 < \delta < 1$, there exists a set $E_\delta \subset [0,1]$ which is perfect, nowhere dense and $|E_\delta| = 1 - \delta$.

Hint: the construction is similar to the construction of the Cantor set, except that at the k–th stage each interval removed has length $\delta 3^{-k}$.

Problem 3.11 Let $\{I_1, \cdots, I_N\}$ be a finite family of open intervals in $\mathbb{R}$ such that $\mathbb{Q} \cap [0,1] \subset \cup_{j=1}^{N} I_j$. Prove that $\sum_{j=1}^{N} |I_j| \geq 1$. Is this true if the family of intervals is infinite?

Problem 3.12 Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of measurable sets in $\mathbb{R}^n$ such that $|E_j \cap E_i| = 0$ for $j \neq i$. Prove that

$$|\bigcup_{j=1}^{\infty} E_j| = \sum_{j=1}^{\infty} |E_j|.$$

Problem 3.13 Let $A, B \subset \mathbb{R}^n$ such that A is measurable. If $A \cap B = \emptyset$ then $|A \cup B|_e = |A| + |B|_e$.

Hint: given $\epsilon > 0$ there exists F closed such that $F \subset A$ and $|A \setminus F| < \epsilon$.

Problem 3.14 Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of measurable sets in $\mathbb{R}^n$. Prove that

1. $\liminf_{j\to\infty} E_j$ and $\limsup_{j\to\infty} E_j$ are measurable.
2. $|\liminf_{j\to\infty} E_j| \leq \liminf_{j\to\infty} |E_j|$.
3. If $|\cup_{j\geq k} E_j| < \infty$ for some k then $|\limsup_{j\to\infty} E_j| \geq \limsup_{j\to\infty} |E_j|$.

Problem 3.15 Let $A \subset \mathbb{R}$ measurable with $|A| < \infty$. Show that the function $f(x) = |(-\infty, x) \cap A|$ is nondecreasing, bounded and uniformly continuous in $\mathbb{R}$.

Problem 3.16 Show that $\mathbb{Q}$ is not a set of type G_δ. Construct a set G of type G_δ such that $\mathbb{Q} \subset G$ and $|G| = 0$.

Problem 3.17 Let E be a measurable set in $\mathbb{R}$ with positive measure. We say that $x \in \mathbb{R}$ is a point of positive measure with respect to E if $|E \cap I| > 0$ for each open interval I containing x. Let $E_+ = \{x \in \mathbb{R} : x \text{ is of positive measure with respect to } E\}$. Prove that

1. E_+ is perfect.
2. $|E \setminus E_+| = 0$.

Problem 3.18 Let E be the set of numbers in $[0, 1]$ whose binary expansion has 0 in all the even places. Show that $|E| = 0$.

Problem 3.19 Let f_k be measurable and $f_k \to f$ a.e. in $\mathbb{R}^n$. Prove that there exists a sequence of measurable sets $\{E_j\}_{j=1}^{\infty}$ such that $|\mathbb{R}^n \setminus \cup_{j=1}^{\infty} E_j| = 0$ and $f_k \to f$ uniformly on each E_j.

Problem 3.20 Let $E \subset \mathbb{R}^n$ be a measurable set with positive measure, and let $D \subset \mathbb{R}^n$ be a countable dense set. Prove that $|\mathbb{R}^n \setminus \cup_{q_k \in D}(q_k + E)| = 0$.

Problem 3.21 An ellipsoid in $\mathbb{R}^n$ with center at x_0 is a set of the form

$$E = \{x \in \mathbb{R}^n : \langle A(x - x_0), x - x_0\rangle \leq 1\},$$

where A is an $n \times n$ positive definite symmetric matrix and $\langle\, ,\rangle$ denotes the Euclidean inner product. Prove that

$$|E| = \frac{\omega_n}{(\det A)^{1/2}},$$

where ω_n is the volume of the unit ball.

Hint: formula of change of variables.

Problem 3.22 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(x, y) = (x + f(x + y), y - f(x + y)).$$

Prove that $|\Phi(E)| = |E|$ for each measurable set $E \subset \mathbb{R}^2$.

Hint: formula of change of variables.

Problem 3.23 Let $A, B \subset \mathbb{R}^n$ such that A is Lebesgue measurable. If $A \cap B = \emptyset$, then

$$|A \cup B|_e = |A| + |B|_e.$$

Here $|\cdot|$ and $|\cdot|_e$ denote Lebesgue and Lebesgue outer measure respectively.

Problem 3.24 Let $E \subset \mathbb{R}^n$ be a measurable subset with positive measure. Prove that $E + E = \{x + y : x, y \in E\}$ contains an open n-dimensional interval containing the origin.

HINT: the function $\chi_E \star \chi_E$ is continuous and positive at the origin.

Problem 3.25 A Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ is a set $\mathcal{F} \subset \mathbb{R}$ such that for each $x \in \mathbb{R}$, $x \neq 0$, there exist unique $q_1, \cdots, q_N \in \mathbb{Q}$ such that $x = \sum_{i=1}^{N} q_i w_i$ with $w_i \in \mathcal{F}$. The existence of $\mathcal{F}$ follows from the axiom of choice via Zorn's lemma.

Let $y \in \mathcal{F}$ and define

$$V = \{\sum q_i w_i : \text{the sum has a finite number of terms, } q_i \in \mathbb{Q}, \text{ and } w_i \neq y,\ w_i \in \mathcal{F}\}.$$

Prove that

1. $\{V + q_j y\}_{q_j \in \mathbb{Q}}$ is a disjoint family and $\mathbb{R} = \cup_{j=1}^{\infty}(V + q_j y)$.
2. The set V is non measurable. HINT: if V were measurable and $|V| > 0$, then by a theorem of Steinhaus, Problem 3.24, $V + V$ would contain an open interval I around the origin. That is $qy \in I$ for $q \in \mathbb{Q}$ sufficiently small. This leads to a contradiction. If on the other hand, $|V| = 0$, then from 1 and the invariance of Lebesgue measure by translations, it would follow that $|\mathbb{R}| = 0$.
3. $V + V = V$.
4. Let $E = V \times C \subset \mathbb{R}^2$, where C is the Cantor set in $[0, 1]$. Then E has 2-dimensional Lebesgue measure equal zero.
5. The set $E + E$ is non measurable in $\mathbb{R}^2$. HINT: $E + E = (V + V) \times (C + C)$ but $C + C = [0, 2]$.

Problem 3.26 Let C be the Cantor set in the interval $[0,1]$.

1. Prove that $C + C = [0,2]$.

2. Prove that there exists a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ contained in C.

   HINT: let $F = \{E \subset C : E \text{ is linearly independent over } \mathbb{Q}\}$. The inclusion relation is a partial order in F and every chain in F has an upper bound. Then by Zorn's lemma F contains a maximal element M. We prove that M is a Hamel basis. Let

   $< M >=$ span of $M = \{$finite combinations of points in M with rational coefficients$\}$.

   We claim that $< M >= \mathbb{R}$. Otherwise, $\exists z \in \mathbb{R}, z \neq 0$ with $z \notin < M >$. Then $\exists q \in \mathbb{Q}$ such that $qz \in [0,2]$. Hence $\exists x_1, x_2 \in C$ such that $qz = x_1 + x_2$, i.e., $z = x_1/q + x_2/q$. If $x_1, x_2 \in M$, then $z \in < M >$, contradiction. Thus $x_1 \notin M$ and $x_2 \in M$, or $x_1 \in M$ and $x_2 \notin M$, or $x_1 \notin M$ and $x_2 \notin M$. In the first case $M \cup \{x_1\}$ is linearly independent. Analogously in the second case $M \cup \{x_2\}$ is linearly independent. In the third case, either $M \cup \{x_1\}$ or $M \cup \{x_2\}$ is linearly independent. So in any case M is not maximal, contradiction.

4 Measurable functions

Let E be a measurable set and $f_n : E \to \overline{\mathbb{R}}$ a sequence of measurable functions.

(A) $f_n \to f$ pointwise in E if $f_n(x) \to f(x)$ for all $x \in E$.

(B) $f_n \to f$ pointwise almost everywhere in E if $f_n(x) \to f(x)$ for almost all $x \in E$.

(C) $f_n \to f$ almost uniformly in E if for each $\epsilon > 0$ there exists a measurable set $F \subset E$ such that $|E \setminus F| \leq \epsilon$ and $f_n \to f$ uniformly in F.

(D) $f_n \to f$ in measure in E if for each $\epsilon > 0$, $\lim_{n\to\infty} |\{x \in E : |f_n(x) - f(x)| \geq \epsilon\}| = 0$.

(E) f_n is a Cauchy sequence in measure in E if for each $\epsilon, \delta > 0$, there exists N such that $|\{x \in E : |f_n(x) - f_m(x)| \geq \epsilon\}| \leq \delta$ for all $n, m \geq N$.

$f : E \to \mathbb{R}$ is a measurable function if $f^{-1}((-\infty, a))$ is a Lebesgue measurable set for all $a \in \mathbb{R}$.

$f : E \to \mathbb{R}$ is a Borel measurable function if $f^{-1}((-\infty, a))$ is a Borel set for all $a \in \mathbb{R}$.

Problem 4.1 If $f_n \to f$ almost uniformly in E, then $f_n \to f$ in measure in E.

Problem 4.2 If $f_n \to f$ in measure and $f_n \to g$ in measure, then $f = g$ a.e.

Problem 4.3 If f_n is a Cauchy sequence in measure, then there exists a subsequence f_{n_k} that is a Cauchy sequence almost uniformly.

Problem 4.4 If f_n is a Cauchy sequence in measure, then there exists f measurable such that $f_n \to f$ in measure.

Problem 4.5 Give an example of a sequence f_n that converges in measure but does not converge a.e.

Problem 4.6 If $|E| < \infty$ and $f_n \to f$ a.e. in E, then $f_n \to f$ in measure. Show that this is false if $|E| = \infty$.

Problem 4.7 If $f_n \to f$ in measure, then there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

Problem 4.8 If $f_n \to f$ in measure, then f_n is a Cauchy sequence in measure.

Problem 4.9 Let $f, f_k : E \to \mathbb{R}$ be measurable functions. Prove that

1. If $f_k \to f$ in measure, then any subsequence f_{k_j} contains a subsequence $f_{k_{j_m}} \to f$ a.e. as $m \to \infty$, .
2. Suppose $|E| < \infty$. If any subsequence f_{k_j} contains a subsequence $f_{k_{j_m}} \to f$ a.e. as $m \to \infty$, then $f_k \to f$ in measure.

Hint: for (2) suppose by contradiction that $f_k \not\to f$ in measure. Then there exists $\epsilon_0 > 0$ such that $|\{x \in E : |f_k(x) - f(x)| \geq \epsilon_0\}| \not\to 0$. Hence there is an increasing sequence $k_j \to \infty$ such that $|\{x \in E : |f_{k_j}(x) - f(x)| \geq \epsilon_0\}| \geq r > 0$, for some $r > 0$ and for all j. By hypothesis there is a subsequence $f_{k_{j_m}} \to f$ a.e. as $m \to \infty$. Now use Egorov to get a contradiction.

Problem 4.10 Let $f : E \to \mathbb{R}$ be a measurable function. Prove that if $B \subset \mathbb{R}$ is a Borel set then $f^{-1}(B)$ is measurable.

Hint: consider $\mathcal{A} = \{A \subset \mathbb{R} : f^{-1}(A) \text{ is measurable}\}$ and show that $\mathcal{A}$ is a σ-algebra that contains the open sets of $\mathbb{R}$.

Problem 4.11 If $f : E \to \mathbb{R}$ is a measurable function and $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $g \circ f$ is measurable.

5 Integration

Problem 5.1 Let $f : [a, b] \to \mathbb{R}$ be integrable and nonnegative. Prove that

$$\left(\int_a^b f(x)\,\cos x\,dx\right)^2 + \left(\int_a^b f(x)\,\sin x\,dx\right)^2 \le \left(\int_a^b f(x)\,dx\right)^2.$$

Hint: write $f(x) = \sqrt{f(x)}\,\sqrt{f(x)}\cos x$ and use Schwartz on the left hand side.

Problem 5.2 Show that

$$\lim_{n\to\infty}\int_{[0,1]^n}\frac{x_1^2+\cdots+x_n^2}{x_1+\cdots+x_n}\,dx = 2/3.$$

Hint: see Makarov, prob. 6.21, p.107.

Problem 5.3 Let $f_1 : [0, M] \to \mathbb{R}$ be a bounded function. Define

$$f_{n+1}(x) = \int_0^x f_n(t)\,dt, \qquad n = 1, 2, \cdots .$$

Prove that the series $\sum_{n=2}^{\infty} f_n(x)$ is uniformly convergent on $[0, M]$ and the sum is a continuous function on $[0, M]$.

Problem 5.4 Let $f : \mathbb{R}^n \to \mathbb{R}$ be integrable. Prove that

1. $|\{x \in \mathbb{R}^n : |f(x)| = \infty\}| = 0$.
2. For each $\epsilon > 0$, $|\{x \in \mathbb{R}^n : |f(x)| \ge \epsilon\}| < \infty$.
3. For each $\epsilon > 0$ there exists a compact set K such that $\int_{\mathbb{R}^n\setminus K} |f(x)|\,dx < \epsilon$.
4. For each $\epsilon > 0$ there exist $M \in \mathbb{R}$ and $A \subset \mathbb{R}^n$ measurable such that $|f(x)| \le M$ in A and $\int_{\mathbb{R}^n\setminus A} |f(x)|\,dx < \epsilon$.
5. For each $\epsilon > 0$ there exist $\delta > 0$ such that if $A \subset \mathbb{R}^n$ is measurable and such that $|A| < \delta$ then $\int_A |f(x)|\,dx < \epsilon$.

Problem 5.5 Let $f : E \to \mathbb{R}$ be measurable with $|E| < \infty$.
Then f is integrable if and only if $\sum_m |\{x \in E : |f(x)| \ge m\}| < \infty$.
Hint: use Abel's summation by parts formula,

$$\sum_{k=1}^{N} a_k\,b_k = A_N\,b_N + \sum_{k=1}^{N-1} A_k\,(b_k - b_{k+1}),$$

where $A_k = \sum_{j=1}^{k} a_j$.

Problem 5.6 Let $f, f_k : \mathbb{R}^n \to \mathbb{R}$ be integrable functions such that $f_k \to f$ a.e. Then $\int_{\mathbb{R}^n} |f_k(x) - f(x)|\, dx \to 0$ if and only if $\int_{\mathbb{R}^n} |f_k(x)|\, dx \to \int_{\mathbb{R}^n} |f(x)|\, dx$.

Problem 5.7 Let $P \subset [0,1]$ be a perfect nowhere dense set, i.e., $\left(\overline{P}\right)^\circ = \emptyset$, with positive measure. Show that $\chi_P(x)$ is not Riemann integrable on $[0,1]$ but it is Lebesgue integrable on $[0,1]$.

Problem 5.8 Let $f : \mathbb{R}^n \to \mathbb{R}$ be integrable and $\epsilon > 0$. Prove that

1. There exists $g : \mathbb{R}^n \to \mathbb{R}$ simple and integrable such that $\int_{\mathbb{R}^n} |f(x) - g(x)|\, dx < \epsilon$.
2. There exists $h : \mathbb{R}^n \to \mathbb{R}$ continuous with compact support such that $\int_{\mathbb{R}^n} |f(x) - h(x)|\, dx < \epsilon$.

Problem 5.9 (Fatou in measure) If $f_k \geq 0$ and $f_k \to f$ in measure in E then

$$\int_E f(x)\, dx \leq \liminf_{k\to\infty} \int_E f_k(x)\, dx.$$

Hint: let $A_k = \int_E f_k(x)\, dx$ and $A = \liminf_{k\to\infty} A_k$; there exists a subsequence $A_{k_m} \to A$ as $m \to \infty$. Since $f_{k_m} \to f$ in measure, there exists a subsequence $f_{k_{m_j}} \to f$ a.e. as $j \to \infty$. Now apply Fatou.

Problem 5.10 (Monotone convergence in measure) If $f_k \geq 0$, $f_k \leq f_{k+1}$, and $f_k \to f$ in measure in E then

$$\int_E f(x)\, dx = \lim_{k\to\infty} \int_E f_k(x)\, dx.$$

Problem 5.11 (Lebesgue in measure) If $|f_k| \leq g$ a.e. with g integrable in E, and $f_k \to f$ in measure in E then

$$\int_E f(x)\, dx = \lim_{k\to\infty} \int_E f_k(x)\, dx.$$

Problem 5.12 Let $f, f_k : \mathbb{R}^n \to \mathbb{R}$ be measurable functions such that $f_k \to f$ a.e. and there exists g integrable such that $|f_k(x)| \leq g(x)$ a.e. for all k. Prove that $f_k \to f$ almost uniformly.

Problem 5.13 Let $|E| < \infty$ and let X denote the class of measurable functions in E. If $f, g \in X$ define

$$d(f,g) = \int_E \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$

Show that (X, d) is a metric space and $f_k \to f$ in (X, d) if and only if $f_k \to f$ in measure in E.

Problem 5.14 Let $C \subset [0,1]$ be the Cantor set and $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if } x \in C \\ p, & \text{if } x \in I \subset [0,1] - P,\ I \text{ interval such that } \ell(I) = 3^{-p} \end{cases}$$

Show that f is measurable and $\int_0^1 f(x)\, dx = 3$.

Problem 5.15 Prove that

1. $\displaystyle\int_0^1 \frac{x \sin x}{1 + (nx)^a}\, dx = o(\frac{1}{n})$, if $a > 1$.
2. $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{dx}{(1 + \frac{x}{n})^n x^{1/n}} = 1$

Problem 5.16 Calculate $\displaystyle\lim_{n\to\infty} \int_0^1 f_n(x) dx$ where:

1. $f_n(x) = \dfrac{\log(x+n)}{n} e^{-x} \cos x$
2. $f_n(x) = \dfrac{nx \log x}{1 + n^2x^2}$
3. $f_n(x) = \dfrac{n\sqrt{x}}{1 + n^2x^2}$
4. $f_n(x) = \dfrac{nx}{1 + n^2x^2}$
5. $f_n(x) = \dfrac{n^{3/2}x}{1 + n^2x^2}$
6. $f_n(x) = \dfrac{n^p x^r \log x}{1 + n^2x^2}$, $r > 0$, $p < \min\{2, 1 + r\}$

7. $f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \sin(\frac{x}{n})$ in the interval $(0, \infty)$.

8. $f_n(x) = \dfrac{1 + nx}{(1 + x)^n}$

Problem 5.17 Prove the following identities.

1. $\lim_{n\to\infty} n^{1/\beta} \int_0^1 x^{1/\beta}(1 - x)^n \frac{dx}{x} = \beta \int_0^\infty e^{-u^\beta} du$ if $\beta > 0$.

2. $\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{(\alpha-1)} dx = \int_0^\infty e^{-x} x^{(\alpha-1)} dx$ if $\alpha > 0$.

3. $\lim_{n\to\infty} \int_\alpha^\infty \sqrt{n}\, e^{-nx^2} dx = \int_\alpha^\infty (\lim_{n\to\infty} \sqrt{n}\, e^{-nx^2})\, dx$ if $\alpha > 0$.

4. $\int_0^1 \frac{x^{1/3}}{1 - x} \log\left(\frac{1}{x}\right) dx = 9 \sum_{n=1}^\infty \frac{1}{(3n + 4)^2}$.

5. $\int_0^\infty \frac{\sin t}{e^t - x} dt = \sum_{n=1}^\infty \frac{x^{n-1}}{n^2 + 1}$ for $-1 < x < 1$.

6. $\int_0^1 \log x \, \sin x \, dx = \sum_{n=1}^\infty \frac{(-1)^n}{(2n)(2n)!}$.

Problem 5.18 If f is Riemann integrable on $[a, b]$ and $f(x) = 0$ for $x \in [a, b] \cap \mathbb{Q}$, then $\int_a^b f(x)\, dx = 0$.

Problem 5.19 Show that

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable on $[0, 1]$. Find the value of $\int_0^1 f(x)\, dx$.

Problem 5.20 Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n f(k/n) = \int_0^1 f(x)\, dx.$$

Problem 5.21 Let $f : [0, 1] \to \mathbb{R}$ be continuous, and $f(x) \neq 0$ for each $x \in (0, 1)$. Suppose that

$$f(x)^2 = 2 \int_0^x f(t)\, dt$$

for each $x \in [0, 1]$. Prove that $f(x) \equiv x$.

Problem 5.22 Let f be a measurable function on $[0, 1]$ and let

$$A = \{x \in [0, 1] : f(x) \in \mathbb{Z}\}.$$

Prove that A is measurable and

$$\int_0^1 \left[\cos\left(\pi\, f(x)\right)\right]^{2n}\, dx \to |A|,$$

as $n \to \infty$.

Problem 5.23 Find all the values of p and q such that the integral

$$\int_{x^2+y^2 \leq 1} \frac{1}{x^{2p} + y^{2q}}\, dxdy$$

converges.

Problem 5.24 Let $f_1, \cdots, f_k$ be continuous real valued functions on the interval $[a, b]$. Show that the set $\{f_1, \cdots, f_k\}$ is linearly dependent on $[a, b]$ if and only if the $k \times k$ matrix with entries

$$\langle f_i, f_j \rangle = \int_a^b f_i(x)\, f_j(x)\, dx$$

has determinant zero.

Problem 5.25 Let $f : [0, +\infty) \to \mathbb{R}$ be continuously differentiable with compact support in $[0, +\infty)$; and $0 < a < b < \infty$. Prove that

$$\int_0^\infty \frac{f(b\, x) - f(a\, x)}{x}\, dx = -f(0)\, \ln(b/a).$$

Problem 5.26 Find all the values of p such that the integral

$$\int_0^\infty \int_0^{\pi/2} e^{-x\, y^p}\, \sin x\, dx\, dy$$

converges.

Problem 5.27 Recall that the convolution of two integrable functions f and g is defined by

$$f \star g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy.$$

Let $f \geq 0$ such that $\int_{\mathbb{R}^n} f(x)\, dx = A < 1$. Define the sequence $f_k = f \star \cdots \star f$ where the convolution is performed k times.

Prove that all the f_k's are integrable and $f_k \to 0$ in $L^1(\mathbb{R}^n)$.

Problem 5.28 Let $f \in L^1(0, +\infty)$ be nonnegative. Prove that

$$\frac{1}{n} \int_0^n x\, f(x)\, dx \to 0, \qquad \text{as } n \to \infty.$$

Problem 5.29 Let $f \geq 0$ in R, set $g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$. Show that if $g \in L(R)$ then $f = 0$ a.e.

Problem 5.30 Given $f \in L^2([0,1])$ define

$$Kf(x) = \frac{1}{x^{4/3}} \int_0^x f(t)\, dt.$$

Show that $\|Kf\|_1 \leq C\|f\|_2$ where C is a constant independent of f.

Hint: use Cauchy-Schwartz.

Problem 5.31 Let A be a measurable subset of $[0, 2\pi]$, show that

$$\int_A \cos nx\, dx \to 0, \qquad \text{as} \quad n \to \infty.$$

Problem 5.32 Let f be a non-negative measurable function in [0,1]. Show that the limit

$$\lim_{n\to\infty} \int_0^1 f(x)^n\, dx$$

exists if and only if

$$|\{x \in [0,1] : f(x) > 1\}| = 0.$$

Problem 5.33 Prove that $\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{\sin x}{1 + n\, x^2}\, dx = 0.$

Problem 5.34 (Variant of Fatou) Let $g_n \in L^1(E)$ and $g_n \to g$ in $L^1(E)$. Suppose f_n are measurable functions on E and $f_n \le g_n$ a.e. for each n. Prove that

$$\limsup_{n\to\infty} \int_E f_n \le \int_E \limsup_{n\to\infty} f_n.$$

Problem 5.35 (Variant of Lebesgue) Let $g_n \in L^1(E)$ and $g_n \to g$ in $L^1(E)$. Suppose f_n are measurable functions on E such that $f_n \to f$ a.e. or $f_n \to f$ in measure, and $|f_n| \le g_n$ a.e. for each n. Prove that

$$\int_E |f_n - f| \to 0,$$

as $n \to \infty$, and consequently $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Problem 5.36 Let $|E| < \infty$. Prove that $f_n \to f$ in measure if and only if

$$\int_E \{|f_n - f| \wedge 1\} \to 0$$

as $n \to \infty$.

Problem 5.37 Let $f \in L^1(0,1)$ and suppose that $\lim_{x\to 1^-} f(x) = A < \infty$. Show that

$$\lim_{n\to\infty} n \int_0^1 x^n \, f(x)\, dx = A.$$

6 Solutions Continuity

1.1 Let $y < x$ with y irrational. Write

$$f(x) - f(y) = \sum_{\{n: y \le r_n < x\}} 1/2^n.$$

Given $\epsilon > 0$ there exists N such that $\sum_{n \ge N} 1/2^n < \epsilon$. Let $\delta = \min\{|y - r_k| : 1 \le k \le N\}$. If $x - y < \delta$ and n is such that $y \le r_n < x$, then $n > N$. Because if $n \le N$ and $y \le r_n < x$, then $r_n - y < x - y < \delta$ which is impossible by definition of δ.

If $y \in \mathbb{Q}$, then $y = r_m$ for some m and so $f(x) - f(y) = \sum_{\{n: y \le r_n < x\}} 1/2^n \ge 1/2^m$ for any $x > y$.

We have

$$\begin{aligned}\int_0^1 f(x)\,dx &= \int_0^1 \sum_{n=1}^{\infty} \chi_{A(x)}(n) \frac{1}{2^n}\,dx = \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 \chi_{A(x)}(n)\,dx \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 \chi_{(r_n, +\infty)}(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{2^n} |[0,1] \cap (r_n, +\infty)|.\end{aligned}$$

1.2 Write

$$\sin\sqrt{x + 4\,n^2\,\pi^2} - \sin\sqrt{y + 4\,n^2\,\pi^2} = \cos\sqrt{\xi + 4\,n^2\,\pi^2}\,\frac{1}{2\sqrt{\xi + 4\,n^2\,\pi^2}}\,(x - y),$$

with some ξ between $x + 4\,n^2\,\pi^2$ and $y + 4\,n^2\,\pi^2$. To show the pointwise convergence pick $y = 0$. For (3), $|f_n(z)| = 1$ when $z + 4\,n^2\,\pi^2 = (k + 1/2)\pi$.

1.3 The class is bounded in $C[a,b]$ and equicontinuous.

1.4 If $|f(x)| \le 1$ on $[a,b]$, then $T(f)$ is bounded. Show that $T(f)$ are equicontinuous writing

$$T(f)(x) - T(f)(z) = \int_a^b (-x^2 + z^2) f(y)\,dy + \int_a^b (e^{-x^2} - e^{-z^2})\,e^y\,f(y)\,dy.$$

1.5 The continuity follows from the uniform convergence of the series. Given $\epsilon > 0$ there exists N such that $\sum_{k=N+1}^{\infty} 2^{-k} < \epsilon$. Write

$$f_n(t) = \sum_{k=1}^{N} 2^{-k} g_n(t - r_k) + \sum_{k=N+1}^{\infty} 2^{-k} g_n(t - r_k) = A_n(t) + B_n(t).$$

We have $\operatorname{supp} g_n(\cdot - r_k) \subset [t - 1/2n, t]$. Let $E = \{r_1, r_2, \cdots, r_N\}$. Suppose first that $t \notin E$ and let $\delta = \operatorname{dist}(t, E)$. If $1/2n < \delta$, then $r_k \notin [t - 1/2n, t]$ for all

$1 \leq k \leq N$ and so $A_n(t) = 0$ for $n > 1/2\delta$. If on the other hand $t \in E$, then $\limsup_{n\to\infty} A_n(t) \leq \sum_{k=1}^N 2^{-k} \limsup_{n\to\infty} g_n(t - r_k) = 0$.

Let (a, b) be an interval. There exists $r_m \in (a, b)$ and so $r_m + 1/n \in (a, b)$ for all n sufficiently large. Therefore $\sup_{x\in(a,b)} f_n(x) \geq f_n(r_m + 1/n) \geq 2^{-m} g_n(1/n) = 2^{-m}$ for all n sufficiently large.

1.6 $f_n(x)$ is a Riemann sum and converges to $\int_x^{x+1} f(t)\,dt$ as $n \to \infty$. Let (a, b) be a finite interval. Write

$$f_n(x) - \int_x^{x+1} f(t)\,dt = \sum_{k=0}^{n-1} \int_{x+k/n}^{x+(k+1)/n} (f(x + k/n) - f(t))\,dt.$$

Since f is uniformly continuous on $[a, b+1]$ we have that $|f(x+k/n)-f(t)| < \epsilon$ for all $x \in [a, b]$, $x + k/n \leq t \leq x + (k + 1)/n$, $0 \leq k \leq n - 1$, and for all n sufficiently large. Therefore

$$\sup_{x\in[a,b]} \left| f_n(x) - \int_x^{x+1} f(t)\,dt \right| < \epsilon$$

for all n sufficiently large.

1.7 (a)$\not\Rightarrow$ (b) $f(x) = 1/x$; (b)$\not\Rightarrow$ (a) $g(x) = 1$, $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$.

1.8 By the absolute continuity $f(x) - f(a) = \int_a^x f'(t)\,dt$ and $g(x) - g(a) = \int_a^x g'(t)\,dt$, for all $a \leq x \leq b$, so $f(x) - g(x) = f(a) - g(a)$ for all $a \leq x \leq b$.

1.9 $f(x)^2 - f(y)^2 = (f(x) + f(y))(f(x) - f(y))$ and f is bounded.

1.10 Suppose $0 < x \leq 1$, and $0 < \alpha < 1$. Then by L'Hospital

$$\lim_{y\to x^+} \frac{y \sin(1/y) - x \sin(1/x)}{(y - x)^\alpha} = \lim_{y\to x^+} \frac{1}{\alpha} (y - x)^{1-\alpha} (\sin(1/y) - (1/y) \cos(1/y)) = 0,$$

and similarly for when $y \to x^-$ and when $x = 0$ ($f(0) = 0$). Therefore, for each $x \in [0, 1]$ there exists a ball $B_{r_x}(x)$ and a constant C_x such that $|f(y) - f(x)| \leq C_x\,|y - x|^\alpha$ for all $y \in B_{r_x}(x)$. Select $B_1(x_1), \cdots B_N(x_N)$ a finite subcovering of $[0, 1]$. Given $x, y \in [0, 1]$ we have $x \in B_J$ and $y \in B_K$ for some J, K. Join x and y with points in $B_i \cap B_j$ and use the triangle inequality.

1.11 Let $\{f_n\}$ be a Cauchy sequence. Given $x \in (a, b)$ take the partition $\{a, x, b\}$. We have

$$|f_n(x) - f_m(x)| \leq |(f_n - f_m)(x) - (f_n - f_m)(a))| + |(f_n - f_m)(a)| \leq \|f_n - f_m\|_{BV} \to 0,$$

as $n, m \to \infty$. Thus, for each $x \in [a, b]$, $\{f_n(x)\}$ is a Cauchy sequence. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Let $\Gamma = \{x_0, \cdots, x_N\}$ be a partition of $[a, b]$. Given $\epsilon > 0$ there exists

n_0 such that

$$\sum_{i=1}^{N} |(f_n - f_m)(x_i) - (f_n - f_m)(x_{i-1}))| < \epsilon$$

for each $n, m \geq n_0$. Letting $n \to \infty$ yields

$$\sum_{i=1}^{N} |(f - f_m)(x_i) - (f - f_m)(x_{i-1}))| < \epsilon$$

for each $m \geq n_0$. Hence $\|f - f_m\|_{BV} < \epsilon$ for $m \geq n_0$.

Let f_n be absolutely continuous such that $\|f_n - f\|_{BV} \to 0$. Given $\epsilon > 0$ there exists N such that $\sum_{i=1}^{k} |(f_n - f)(x_i) - (f_n - f)(x_{i-1})| < \epsilon$ for each $n \geq N$ and any partition $\Gamma = \{x_0, \cdots, x_k\}$. Since f_N is absolutely continuous, there exists $\delta > 0$ such that if $(a_1, b_1), \cdots, (a_r, b_r)$ is any collection of disjoint intervals of $[a, b]$ with $\sum_{i=1}^{r}(b_i - a_i) < \delta$, then $\sum_{i=1}^{r} |f_N(b_i) - f_N(a_i)| < \epsilon$. If $\Gamma = \{a, a_1, b_1, a_2, b_2, \cdots, a_r, b_r, b\}$, then

$$\begin{aligned}\sum_{i=1}^{r} |f(b_i) - f(a_i)| &\leq \sum_{i=1}^{k} |(f - f_N)(b_i) - (f - f_N)(a_i)| + \sum_{i=1}^{k} |f_N(b_i) - f_N(a_i)| \\ &\leq \|f - f_N\|_{BV} + \epsilon < 2\,\epsilon.\end{aligned}$$

7 Solutions Semicontinuity

2.6 Suppose $y \in \mathbb{Q}$, $y = r_m$. Then $f(x) \geq \frac{1}{2^m} + f(y)$ for all $x > y$, so

$$\inf_{y<x<y+\delta} f(x) \geq \frac{1}{2^m} + f(y) > f(y)$$

for any $\delta > 0$. We now estimate $\inf\limits_{y-\delta<x\leq y} f(x)$. If $x \leq y$, then

$$f(x) = f(y) - \sum_{\{n:x\leq r_n<y\}} 1/2^n.$$

Suppose $y - \delta < x \leq y$, then $\{n : x \leq r_n < y\} \subset \{n : y - \delta < r_n < y\}$ and so

$$f(x) \geq f(y) - \sum_{\{n:y-\delta<r_n<y\}} 1/2^n.$$

We claim that for each $N \geq 1$ there exists $\delta > 0$ such that if $r_j \in (r_m - \delta, r_m)$ then $j \geq N$. Suppose by contradiction that this is false. Then there exists $N_0 \geq 1$ such that for each $\delta = 1/k$ there exists a $j(k)$ such that $r_{j(k)} \in (r_m - 1/k, r_m)$ and $j(k) < N_0$. Hence $j(k)$ takes only a finite number of values between 1 and N_0, and therefore there exists a subsequence k_i such that $k_i \to \infty$ as $i \to \infty$ with $j(k_i) = s$ and $r_s \in (r_m - 1/k_i, r_m)$ for all i. Letting $i \to \infty$ we get $r_m \leq r_s < r_m$, a contradiction.

Now given $\epsilon > 0$, there exists $N \geq 1$ such that $\sum_{n\geq N} 1/2^n < \epsilon$, and by the claim there exists $\delta_0 > 0$ such that $\{n : y - \delta_0 < r_n < y\} \subset \{n : n \geq N\}$. Consequently,

$$\sum_{\{n:y-\delta_0<r_n<y\}} 1/2^n \leq \sum_{n\geq N} 1/2^n < \epsilon,$$

and so $f(x) \geq f(y) - \epsilon$ for $y - \delta < x \leq y$ and $0 < \delta \leq \delta_0$. Thus,

$$\inf_{y-\delta<x\leq y} f(x) \geq f(y) - \epsilon$$

for all $0 < \delta \leq \delta_0$, and so $\lim\limits_{\delta\to 0} \inf\limits_{y-\delta<x\leq y} f(x) \geq f(y) - \epsilon$ for any $\epsilon > 0$.