Real Analysis II, Prof. Gutiérrez, Abstract measures II
Week of April 15, 2009

Assume $(\mathcal{S}, \Sigma, \mu)$ is a measure space, i.e., Σ is a σ-algebra of sets in $\mathcal{S}$ and μ is a measure on Σ.

1. Let $\phi : \Sigma \to \mathbb{R}$ be an additive set function. Prove that

$$V(E) = \sup\left\{\sum_1^N |\phi(F_i)| : \{F_i\}_i^N \subset \Sigma, \cup_1^N F_i \subset E,\ F_i \text{ are disjoint}\right\}$$

   HINT: let $P \subset E$ be the set in the Hahn decomposition, i.e., $\phi(A) \geq 0$ for all $A \subset P$ and $\phi(A) \leq 0$ for all $A \subset E \setminus P$. First prove that $V(E) \leq |\phi(P)| + |\phi(E \setminus P)|$, which proves one inequality. To prove the other use Jordan's decomposition.

2. Let $(\mathcal{S}, \Sigma, \mu)$ be a finite measure space and $f : \mathcal{S} \to \mathbb{R}$ a nonnegative Σ-measurable function. Prove that $\lim_{k\to\infty} \int_{\mathcal{S}} f^k \, d\mu$ exists if and only if $\mu\{x \in \mathcal{S} : f(x) > 1\} = 0$.

3. Let μ be a regular Borel measure in $\mathbb{R}^n$. Prove that the continuous functions with compact support are dense in $L^p(\mathbb{R}^n, \mu)$, $1 \leq p < \infty$.

   HINT: use that the simple functions that are zero outside a set of finite measure are dense in L^p.

4. Let $(\mathcal{S}, \Sigma, \mu)$ be a finite measure space. Suppose $\|f_n\|_{L^p(\mathcal{S})} \leq M$ and $f_k \to f$ a.e. Prove that

$$\int_{\mathcal{S}} f_k \, g \, d\mu \to \int_{\mathcal{S}} f \, g \, d\mu$$

   for all $g \in L^{p'}(\mathcal{S})$.