

Real Analysis II, Prof. Gutiérrez, Abstract measures I
Week of April 2, 2009

Assume $(\mathcal{S}, \Sigma, \mu)$ is a measure space, i.e., Σ is a σ -algebra of sets in \mathcal{S} and μ is a measure on Σ .

1. Prove that the following triplets $(\mathcal{S}, \Sigma, \mu)$ are measure spaces.

1. \mathcal{S} is an infinite set, $\Sigma = \mathcal{P}(\mathcal{S})$, and $\mu(E) = \#(E)$ if E is finite and $\mu(E) = +\infty$ if E is infinite.
2. $\mathcal{S} = [0, 1]$, $\Sigma = \mathcal{P}([0, 1])$, $\mu(E) = 0$ if E is of the first category, and $\mu(E) = +\infty$ if E is of the second category.
3. \mathcal{S} is an infinite set, $\Sigma = \mathcal{P}(\mathcal{S})$. Let x_1, x_2, \dots be a fix sequence of distinct points in \mathcal{S} ; for $E \subset \mathcal{S}$ set $\mu(E) = \sum_{x_i \in E} \frac{1}{2^i}$.

2. Let $E, F \in \Sigma$. Prove that $\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \cup F)$.

3. Given $E, F \in \Sigma$ let $\rho(E, F) = \mu(E \Delta F)$. Prove that $\rho(E, F) \geq 0$; $\rho(E, F) = \rho(F, E)$; and $\rho(E, F) \leq \rho(E, G) + \rho(G, F)$.

4. (Borel-Cantelli) Let $E_k \in \Sigma, k = 1, 2, \dots$. Suppose that $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Prove that $\mu(\limsup_k E_k) = 0$.

5. Let $f \in L(\mathbb{R}^n)$ and for a Lebesgue measurable set $E \subset \mathbb{R}^n$ define $\phi(E) = \int_E f(x) dx$. Show that ϕ is an additive set function on the σ -algebra of Lebesgue measurable sets.

Let \bar{V} and \underline{V} be the upper and lower variation of ϕ . Show that

$$\bar{V}(E) = \int_E f^+(x) dx \quad \text{and} \quad \underline{V}(E) = \int_E f^-(x) dx.$$

6. Let $(\mathcal{S}, \Sigma, \mu)$ be a σ -finite measure space, that is, $\mathcal{S} = \cup_{k=1}^{\infty} E_k$ with $\mu(E_k) < \infty$. Suppose that $\mu(\mathcal{S}) = +\infty$. Show that given $t > 0$ there exists $E \in \Sigma$ such that $t \leq \mu(E) < +\infty$.