Real Analysis, Math 558, Prof. Gutiérrez
Miscellaneous Problems
Week of March 19, 2009

Notation.

- Let $\Omega \subset \mathbb{R}^n$ be open. The function $f : \Omega \to \overline{\mathbb{R}}$ is lower (upper) semicontinuous if $f(z) \leq \liminf_{x\to z} f(x) = \lim_{\delta\to 0} \inf_{|x-z|<\delta} f(x)$ ($f(z) \geq \limsup_{x\to z} f(x) = \lim_{\delta\to 0} \sup_{|x-z|<\delta} f(x)$).
- The Hardy-Littlewood maximal function of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is

$$f^*(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\, dy.$$

1. Prove that the following statements are equivalent:
    1. f is lower semicontinuous;
    2. $f^{-1}(c, +\infty)$ is open for each $c \in \mathbb{R}$;
    3. $f^{-1}(-\infty, c]$ is closed for each $c \in \mathbb{R}$;
    4. If $x_k \to z$, then $f(z) \leq \sup_k f(x_k)$;
    5. For each $z \in \Omega$ and for each $\epsilon > 0$ there exists a neighborhood V of z such that $f(x) \geq f(z) - \epsilon$ for each $x \in V$.
2. Let $g : \Omega \to \mathbb{R}$ and define
$$f(x) = \liminf_{z\to x} g(z).$$
Prove that f is lower semicontinuous.
3. Prove that if $E \subset \mathbb{R}^n$, then χ_E is lower semicontinuous if and only if E is open.
4. Let $\{f_\alpha\}_{\alpha\in J}$ be a family of lower semicontinuous function in Ω. Prove that $f(x) = \sup_{\alpha\in J} f_\alpha(x)$ is lower semicontinuous.
5. Prove that the Hardy-Littlewood maximal function is lower semicontinuous.
6. The "non-centered" maximal function is given by
$$f^{**}(x) = \sup\left\{ \frac{1}{|B|} \int_B |f(y)|\, dy : \text{where } B \text{ is any ball containing } x \right\}.$$
Prove that there are positive constants C_1, C_2 independent of f such that $C_1 f^*(x) \leq f^{**}(x) \leq C_2 f^*(x)$, that is, these two maximal functions are always comparable.
7. Let f be a continuous function in [-1,2]. Given $0 \leq x \leq 1$, and $n \geq 1$ define the sequence of functions
$$f_n(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\, dt.$$
Show that f_n is continuous in [0,1] and f_n converges uniformly to f in [0,1].
8. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a uniformly bounded sequence of functions. Show that for each countable subset $S \subset R$ there exists a subsequence of f_n which converges in S.

    Hint: select the subsequence by using a diagonal process

9. Let $f_n(x) = \cos(n\,x)$ on $\mathbb{R}$. Prove that there is no subsequence f_{n_k} converging uniformly in $\mathbb{R}$. HINT: if there is uniformly convergent subsequence to a limit f, then show that f has integral zero over each finite interval and therefore $f = 0$. Hence $\sup_{\mathbb{R}} \cos(n_k\,x) \to 0$ as $k \to \infty$, which is impossible.

10. Let $f_1, \cdots, f_k$ be continuous real valued functions on the interval $[a, b]$. Show that the set $\{f_1, \cdots, f_k\}$ is linearly dependent on $[a, b]$ if and only if the $k \times k$ matrix with entries

$$\langle f_i, f_j \rangle = \int_a^b f_i(x)\, f_j(x)\, dx$$

has determinant zero.

11. Recall that the convolution of two integrable functions f and g is defined by

$$f \star g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy.$$

Let $f \geq 0$ such that $\int_{\mathbb{R}^n} f(x)\, dx = A < 1$. Define the sequence $f_k = f \star \cdots \star f$ where the convolution is performed k times.

Prove that all the f_k's are integrable and $f_k \to 0$ in $L^1(\mathbb{R}^n)$.

12. If $f \in L(\mathbb{R}^n)$ and f is not identically zero, then there exists a constant $C > 0$ such that $f^*(x) \geq \dfrac{C}{|x|^n}$ for all $|x| \geq 1$. Conclude that $f^* \notin L(\mathbb{R}^n)$.