$H_{\ln 2/\ln 3}(C) = 1$ FOR C THE CANTOR SET
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Lemma 0.1. *Let* $0 < \alpha < 1$*,* $a > 0$*, and*

$$f(x,y) = \frac{x^\alpha + y^\alpha}{(x+a+y)^\alpha}.$$

Then $\max_{(x,y)\in[0,a]^2} f(x,y) = \dfrac{2}{3^\alpha}$*, and so if* $\alpha = \dfrac{\ln 2}{\ln 3}$*, we get*

$$x^\alpha + y^\alpha \leq (x+a+y)^\alpha, \quad \textit{for} \quad 0 \leq x \leq a, 0 \leq y \leq a.$$

Therefore, if I *is an interval such that* $I = I_1 \cup I_2 \cup I_3$ *with* I_j *intervals such that* $|I_1|, |I_3| \leq |I_2|$ *then*

$$|I_1|^\alpha + |I_3|^\alpha \leq |I|^\alpha.$$

Proof. Fix $0 < y \leq a$ and let $g(x) = f(x,y)$. We have $g'(x) \geq 0$ if and only if $x \leq \left(\dfrac{y+a}{y^\alpha}\right)^{1/(1-\alpha)}$. Since $0 < y \leq a$, we have $\left(\dfrac{y+a}{y^\alpha}\right)^{1/(1-\alpha)} \geq a$, and so g is non decreasing in the interval $[0,a]$ and therefore $\max_{x\in[0,a]} g(x) = g(a) = f(a,y)$. Let $h(y) = f(a,y)$. We have $h'(y) \geq 0$ if and only if $2a^{1-\alpha} \geq y^{1-\alpha}$, and so h is non decreasing in the interval $0 \leq y \leq a$, and therefore $\max_{y\in[0,a]} h(y) = h(a) = f(a,a) = 2/3^\alpha$. □

Given $A \subset \mathbb{R}$, the Hausdorff measure of dimension α of A is

$$H_\alpha(A) = \lim_{\epsilon\to 0} \inf \left\{ \sum_k |I_k|^\alpha : A \subset \cup_{k=1}^\infty I_k,\ I_k \text{ intervals },|I_k| \leq \epsilon \right\}.$$

If C is the Cantor set in $[0,1]$, we will prove that $H_{\ln 2/\ln 3}(C) = 1$. Let $\{I_m\}_{m=1}^\infty$ be a covering of C by open intervals and since C is compact let $\beta > 0$ be the

Lebesgue number of the covering,[*] that is, given $x \in C$ there exists m such that $[x-\beta, x+\beta] \subset I_m$. Recall that $C = \cap_{k=1}^{\infty} C_k$ with $C_k = \cup_{j=1}^{2^k} J_j^k$ and J_j^k are closed pairwise disjoint intervals all of length $1/3^k$. Therefore, if $1/3^k < \beta$, then given J_j^k there exists a I_m such that $J_j^k \subset I_m$. Given m, let $S_m = \{j : J_j^k \subset I_m\}$. The intervals J_j^k with $j \in S_m$ are all consecutive and let I_m^* be the smallest interval containing all of them. Between two consecutive intervals J_j^k with $j \in S_m$ there is an interval J^* that was removed in the construction of C_k. Take the largest of these intervals removed, that is, let J^{**} be the longest interval removed in the construction of C_k and contained in I_m^*. We then have that $I_m^* = J_1 \cup J^{**} \cup J_2$. By construction of C_k we have that $|J_1|, |J_2| \leq |J^{**}|$. We have by the lemma that

$$|I_m|^\alpha \geq |I_m^*|^\alpha \geq |J_1|^\alpha + |J_2|^\alpha.$$

Once again, pick the largest interval removed in the construction of C_k that is contained in J_1, say this interval is J^{***}. We again have $J_1 = J_1^1 \cup J^{***} \cup J_2^1$ and $|J_1^1|, |J_2^1| \leq |J^{***}|$. Similarly, $J_2 = J_1^2 \cup J^{****} \cup J_2^2$ and $|J_1^2|, |J_2^2| \leq |J^{****}|$. Therefore applying the lemma once again

$$|I_m|^\alpha \geq |I_m^*|^\alpha \geq |J_1|^\alpha + |J_2|^\alpha \geq |J_1^1|^\alpha + |J_2^1|^\alpha + |J_1^2|^\alpha + |J_2^2|^\alpha.$$

Continuing in this way we exhaust all I_m^* obtaining

$$|I_m|^\alpha \geq |I_m^*|^\alpha \geq \sum_{j \in S_m} |J_j^k|^\alpha.$$

Therefore

$$\sum_{m=1}^{\infty} |I_m|^\alpha \geq \sum_{m=1}^{\infty} |I_m^*|^\alpha \geq \sum_{m=1}^{\infty} \sum_{j \in S_m} |J_j^k|^\alpha \geq \sum_{j=1}^{2^k} |J_j^k|^\alpha = 2^k \left(\frac{1}{3^k}\right)^\alpha = 1,$$

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[*] If E is a compact metric space and $\{O_\lambda\}_{\lambda \in \Lambda}$ is a open cover of E, then there exists a positive number β (the Lebesgue number) such that for each ball $B_\beta(x)$ with $x \in E$ there exists $\lambda \in \Lambda$ such that $B_\beta(x) \subset O_\lambda$. To prove this, since the O_λ's are open and cover E, for each $x \in E$ there exists λ such that $x \in O_\lambda$ and so there exists $r_x > 0$ such that the ball $B_{r_x}(x) \subset O_\lambda$. Consider the open cover $\{B_{r_x/2}(x)\}_{x \in E}$. By compactness there is a finite sub-cover $B_{r_1/2}(x_1), \cdots, B_{r_N/2}(x_N)$ of E. Let $\beta = \min_{1 \leq j \leq N} r_j/2$. Prove that β has the desired property.

since $\alpha = \ln 2/\ln 3$. Hence $H_{\ln 2/\ln 3}(C) \geq 1$. To prove the opposite inequality we have $H_\alpha(C) \leq H_\alpha(C_k) \leq \sum_{j=1}^{2^k} |J_j^k|^\alpha = 1$.

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