

Real Analysis II, Prof. Gutiérrez, Problems on Fubini
Week of January 20, 2009

1. Let $f \in L(\mathbb{R})$ and for $h > 0$ define

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) dt.$$

Show that $\|f_h\|_1 \leq \|f\|_1$.

2. Let f be a non-negative measurable function in \mathbb{R} . Suppose that

$$\iint_{\mathbb{R}^2} f(4x)f(x-3y) dx dy = 2.$$

Calculate $\int_{-\infty}^{\infty} f(x) dx$.

3. Let $g \in L(\mathbb{R})$. Determine all values of α for which the function

$$\phi(x, t) = |x|^{-\alpha} t^{\alpha-1} g(x) \chi_{[-t, t]}(x),$$

is integrable in $\mathbb{R} \times \mathbb{R}^+$.

4. Given $f \in L^2([0, 1])$ define

$$Kf(x) = \frac{1}{x^{4/3}} \int_0^x f(t) dt.$$

Show that $\|Kf\|_1 \leq C\|f\|_2$ where C is a constant independent of f .

Hint: use Cauchy-Schwartz.

5. Let $f \in L^p(E)$, $p > 0$. Show that

$$\int_E |f|^p dx = p \int_0^{\infty} t^{p-1} |\{x \in E : |f(x)| > t\}| dt.$$

6. Determine all values of $p \in \mathbb{R}$ such that the function

$$f(x, y) = \frac{1}{(x^2 + y^4)^p}$$

is integrable on the square $[-1, 1] \times [-1, 1]$.

Hint: consider the region $[0, 1] \times [0, 1]$, make the change of variables $x' = x$, $y = \sqrt{y'}$, and next change to polar coordinates.