A COVERING LEMMA

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Definition 1. *Let X be a set. A partial order in X is a relation $\preccurlyeq$ such that*

(1) $x \preccurlyeq x \ \forall x \in X$;

(2) *if* $x \preccurlyeq y$ *and* $y \preccurlyeq x \Rightarrow x = y$;

(3) *if* $x \preccurlyeq y$ *and* $y \preccurlyeq z \Rightarrow x \preccurlyeq z$.

We say $(X, \preccurlyeq)$ *is a set partially ordered.*

Definition 2. *Let* $(X, \preccurlyeq)$ *be partially ordered. The subset* $Y \subset X$ *is a chain or totally ordered if* $\forall x, y \in Y$ *then either* $x \preccurlyeq y$ *or* $y \preccurlyeq x$. *An upper bound of* Y *is an element* $u \in X$ *such that* $y \preccurlyeq u$ *for all* $y \in Y$.

Theorem 3 (Zorn's lemma)**.** [1] *Let* $(X, \preccurlyeq)$ *be partially ordered and suppose that every chain in* X *has an upper bound. Then* X *has a maximal element, that is, there exists* $m \in X$ *such that if* $m \preccurlyeq x$, *then* $x = m$.

Corollary 4. *Let* $\mathcal{C}$ *be a family of balls in a quasi metric space*[2]. *Then there exists a subfamily* $\mathcal{F}$ *of pairwise disjoint balls that is maximal, that is, if* $B \in \mathcal{C} \setminus \mathcal{F}$, *then there exists* $B' \in \mathcal{F}$ *such that* $B \cap B' \neq \emptyset$.

Proof. Define the class

$$X = \{\{B_\alpha\}_{\alpha \in I} : B_\alpha \in \mathcal{C} \text{ and } B_\alpha \text{ are all disjoint}\},$$

that is, X is the collection of all families of balls from $\mathcal{C}$ that are pairwise disjoint. We define: if $\mathcal{F}_1, \mathcal{F}_2 \in X$ then $\mathcal{F}_1 \preccurlyeq \mathcal{F}_2$ iff $\mathcal{F}_1 \subset \mathcal{F}_2$. $(X, \preccurlyeq)$ is a set partially ordered.

[1] Zorn's lemma is equivalent to the axiom of choice.

[2] The triangle inequality holds with a constant, that is, $d(x, y) \leq K(d(x, z) + d(z, y))$.

Let $Y \subset X$ be a chain and let $\mathcal{F} = \cup_{\mathcal{F}' \in Y} \mathcal{F}'$. It is clear that $\mathcal{F}$ is an upper bound for Y, we only have to prove that $\mathcal{F} \in X$. Let $B, B' \in \mathcal{F}$ with $B \neq B'$. Then $B \in \mathcal{F}'$ and $B' \in \mathcal{F}''$, with $\mathcal{F}', \mathcal{F}'' \in Y$. Since Y is a chain, either $\mathcal{F}' \preccurlyeq \mathcal{F}''$ or $\mathcal{F}'' \preccurlyeq \mathcal{F}'$. In the first case $B, B' \in \mathcal{F}''$ and in the second case $B, B' \in \mathcal{F}'$, so, in any case, $B \cap B' = \emptyset$. Therefore by Zorn's lemma, X has a maximal element $\mathcal{F}_0$ and the corollary is proved. □

We can now prove the following covering lemma.

Lemma 5. *Let C be a family of balls in a quasi metric space and suppose that*

$$\sup\{\text{diam}(B) : B \in C\} < \infty.$$

Then there exists a subfamily[3] $\mathcal{F} \subset C$ of disjoint balls such that

$$\bigcup_{B \in C} B \subset \bigcup_{B \in \mathcal{F}} K(4K+1)B,$$

where αB is the ball with the same center as B and with radius α times the radius of B, K being the constant in the quasi triangle inequality.

Proof. Let $d = \sup\{\text{radius}(B) : B \in C\}$ and $C_j = \{B \in C : d/2^j < \text{radius}(B) \leq d/2^{j-1}\}$ with $j = 1, 2, \cdots$. Applying Corollary 4 to C_1, let $\mathcal{F}_1$ be a maximal subfamily of disjoint balls in C_1. If the families $\mathcal{F}_j$ with $j = 1, \cdots, k-1$ have been selected we let

$$\tilde{C}_k = \{B \in C_k : B \cap B' = \emptyset \quad \forall B' \in \cup_{j=1}^{k-1} \mathcal{F}_j\},$$

and once again from Corollary 4, let $\mathcal{F}_j$ be a maximal subfamily of disjoint balls in $\tilde{C}_k$.

We let $\mathcal{F} = \cup_{j=1}^{\infty} \mathcal{F}_j$ and we shall prove $\mathcal{F}$ has the desired properties. If $B \in C$, then there exists j such that $B \in C_j$. There are two possibilites: $B \in \tilde{C}_j$ or $B \notin \tilde{C}_j$.

(i) if $B \in \tilde{C}_j$ and since $\mathcal{F}_j$ is maximal in $\tilde{C}_j$, then there exists $B' \in \mathcal{F}_j$ such that $B \cap B' \neq \emptyset$.

[3]If for example the quasi metric space is separable, then every family of disjoint balls is at most countable.

(ii) if $B \notin \tilde{C}_j$ and since $B \in C_j$, then there exists $B' \in \cup_{k=1}^{j-1} \mathcal{F}_k$ such that $B \cap B' \neq \emptyset$. If (i), then radius$(B') \geq d2^{-j}$. If (ii), then radius$(B') \geq d2^{-j+1}$. So in any case, there exists a ball $B' \in \cup_{k=1}^{j} \mathcal{F}_k$ such that radius$(B') \geq d2^{-j}$ and $B \cap B' \neq \emptyset$. We claim that $B \subset K(4K+1)B'$. Indeed, if $B = B_r(x)$ and $B' = B_{r'}(y)$, then $r < 2r'$ and let $z \in B \cap B'$. If $w \in B$, then $d(w,y) \leq K(d(w,z) + d(z,y)) \leq K(K(d(w,x) + d(x,z)) + d(z,y)) \leq K(K(4r') + r') \leq K(4K+1)r'$ and we are done.

□

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