

SOLUTIONS TO PROBS 16 AND 17

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Problem 16. Let $f : [a, b] \rightarrow \mathbb{R}$ and define

$$\bar{M} = \sup_{x \in [a, b]} |f(x)|; \quad M = \sup_{x \in [a, b]} f(x); \quad \bar{m} = \inf_{x \in [a, b]} |f(x)|; \quad m = \inf_{x \in [a, b]} f(x).$$

Then

$$\bar{M} - \bar{m} \leq M - m.$$

In fact, there exists $\xi \in [a, b]$ such that $\bar{M} < |f(\xi)| + \epsilon$ and there exists $\zeta \in [a, b]$ such that $\bar{m} > |f(\zeta)| - \epsilon$. Hence

$$\bar{M} - \bar{m} < |f(\xi)| - |f(\zeta)| + 2\epsilon \leq |f(\xi) - f(\zeta)| + 2\epsilon,$$

and since $|f(\xi) - f(\zeta)| \leq M - m$ we obtain

$$\bar{M} - \bar{m} \leq M - m + 2\epsilon$$

for all $\epsilon > 0$ and the claim is proved.

From the claim it follows immediately that

$$0 \leq U_\Gamma(|f|) - L_\Gamma(|f|) \leq U_\Gamma(f) - L_\Gamma(f)$$

for each partition Γ of $[a, b]$ where U_Γ and L_Γ are the upper and lower Riemann-Stieltjes sums respectively.

Problem 17. Prove first the following. Let $f \in C[a, b]$, $\phi \in BV[a, b]$, and define the oscillation of f by

$$\omega(f, \delta) = \sup\{|f(x) - f(y)| : |x - y| \leq \delta\}$$

Let $\Delta = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ with $\|\Delta\| < \delta$ and $\xi_i \in [x_{i-1}, x_i]$. Then

$$\left| \sum_{i=1}^n f(\xi_i)(\phi(x_i) - \phi(x_{i-1})) - \int_a^b f d\phi \right| \leq \omega(f, \delta) V_a^b \phi.$$

Next to solve Problem 11 let $\epsilon_n = \sup\{|f_n(x) - f(x)| : x \in [a, b]\}$. Write

$$\begin{aligned} \left| \int_a^b f_n d\phi_n - \int_a^b f d\phi \right| &= \left| \int_a^b (f_n - f) d\phi_n - \int_a^b f d(\phi_n - \phi) \right| \\ &\leq \epsilon_n V_a^b \phi_n + \left| \int_a^b f d\phi_n - \int_a^b f d\phi \right|. \end{aligned}$$

Let $\epsilon > 0$ and choose $\delta > 0$ such that $\omega(f, \delta) \leq \epsilon$ and let $\Delta = \{x_0, x_1, \dots, x_m\}$ be a partition of $[a, b]$ with $\|\Delta\| < \delta$. From the first statement we have

$$\left| \sum_{i=1}^m f(\xi_i)(\phi_n(x_i) - \phi_n(x_{i-1})) - \int_a^b f d\phi_n \right| \leq \omega(f, \delta) V_a^b \phi_n \leq \epsilon M;$$

and also

$$\left| \sum_{i=1}^m f(\xi_i)(\phi(x_i) - \phi(x_{i-1})) - \int_a^b f d\phi \right| \leq \omega(f, \delta) V_a^b \phi \leq \epsilon M.$$

Since $\phi_n \rightarrow \phi$ pointwise in $[a, b]$, there exists N such that

$$\left| \sum_{i=1}^m f(\xi_i)(\phi_n(x_i) - \phi_n(x_{i-1})) - \sum_{i=1}^m f(\xi_i)(\phi(x_i) - \phi(x_{i-1})) \right| \leq \epsilon$$

for all $n > N$. Hence

$$\left| \int_a^b f d\phi_n - \int_a^b f d\phi \right| \leq \epsilon(2M + 1)$$

for all $n > N$ and we are done.

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