

# Real Analysis Problems

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September 14, 2009

# 1 Continuity

**Problem 1.1** Let  $r_n$  be the sequence of rational numbers and

$$f(x) = \sum_{\{n:r_n < x\}} \frac{1}{2^n}.$$

Prove that

1.  $f$  is continuous on the irrationals.
2.  $f$  is discontinuous on the rationals.
3. Calculate  $\int_0^1 f(x) dx$ .

Hint: for (3) set  $A(x) = \{n : r_n < x\}$  so  $f(x) = \sum_{n=1}^{\infty} \chi_{A(x)}(n) \frac{1}{2^n}$ ; use Fubini.

**Problem 1.2** Let  $f_n(x) = \sin \sqrt{x + 4n^2 \pi^2}$  on  $[0, +\infty)$ . Prove that

1.  $f_n$  is equicontinuous on  $[0, +\infty)$ .
2.  $f_n$  is uniformly bounded.
3.  $f_n \rightarrow 0$  pointwise on  $[0, +\infty)$ .
4. There is no subsequence of  $f_n$  that converges to 0 uniformly.
5. Compare with Arzelà-Ascoli.

**Problem 1.3** Prove that the class of Lipschitz functions  $f$  in  $[a, b]$  with Lipschitz constant  $\leq K$  and  $f(a) = 0$  is a compact set in  $C([a, b])$ .

**Problem 1.4** Let  $B$  be the unit ball in  $C([a, b])$ . Define for  $f \in C([a, b])$

$$Tf(x) = \int_a^b \left( -x^2 + e^{-x^2+y} \right) f(y) dy.$$

Prove that  $T(B)$  is relatively compact in  $C[a, b]$ .

**Problem 1.5** Let  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  be the continuous function that is zero outside the interval  $[0, 2/n]$ ,  $g_n(1/n) = 1$ , and  $g_n$  is linear on  $(0, 1/n)$  and  $(1/n, 2/n)$ . Prove that  $g_n \rightarrow 0$  pointwise in  $\mathbb{R}$  but the convergence is not uniform on any interval containing 0. Let  $r_k$  be the rational numbers and define

$$f_n(t) = \sum_{k=1}^{\infty} 2^{-k} g_n(t - r_k).$$

Prove that

1.  $f_n$  is continuous on  $\mathbb{R}$ .
2.  $f_n \rightarrow 0$  pointwise in  $\mathbb{R}$ .
3.  $f_n$  does not converge uniformly on any interval of  $\mathbb{R}$ .

**Problem 1.6** Let  $f \in C(\mathbb{R})$  and  $f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right)$ . Show that  $f_n$  converges uniformly on every finite interval.

**Problem 1.7** Consider the following statements:

- (a)  $f$  is a continuous function a.e. on  $[0, 1]$
- (b) there exists  $g$  continuous on  $[0, 1]$  such that  $g = f$  a.e.

Show that (a) does not imply (b), and (b) does not imply (a).

**Problem 1.8** Show that if  $f$  and  $g$  are absolutely continuous functions in  $[a, b]$  and  $f'(x) = g'(x)$  a.e., then  $f(x) - g(x) = \text{constant}$ , for each  $x \in [a, b]$ .

**Problem 1.9** If  $f$  is absolutely continuous in  $[0, 1]$ , then  $f^2$  is absolutely continuous in  $[0, 1]$ .

**Problem 1.10** Let  $0 < \alpha \leq 1$ . A function  $f \in C^\alpha([0, 1])$ ,  $f$  is Hölder continuous of order  $\alpha$ , if there exists  $K \geq 0$  such that  $|f(x) - f(y)| \leq K|x - y|^\alpha$  for all  $x, y \in [0, 1]$ . Prove that

1.  $C^\alpha[0, 1] \subset C^\beta([0, 1])$  for  $0 < \beta \leq \alpha \leq 1$ .
2. If  $f$  is Hölder continuous of order one, that is,  $f$  is Lipschitz, then  $f$  is absolutely continuous on  $[0, 1]$ .

3. Let  $a > 0$ . Define  $g(x) = x^a \sin(x^{-a})$  for  $x \neq 0$  and  $f(0) = 0$ .
4. Prove that  $g \notin BV[0, 1]$ . Prove that  $g \in C^\alpha([0, 1])$  with  $\alpha = a/(1 + a)$ , and  $g \notin C^\beta([0, 1])$  for  $\beta > a/(1 + a)$ .

**Problem 1.11** Consider the space  $BV[a, b]$  of bounded variation functions in  $[a, b]$  with the norm

$$\|f\|_{BV} = \sup_{\Gamma} \sum_{i=1}^k |f(\alpha_i) - f(\alpha_{i-1})| + |f(a)|,$$

where the supremum is taken over all partitions  $\Gamma = \{\alpha_0, \dots, \alpha_k\}$  of  $[a, b]$ . Prove that

1.  $(BV[a, b], \|\cdot\|)$  is a Banach space;
2. the set of absolutely continuous functions in  $[a, b]$  is a closed subspace of  $BV[a, b]$ .

**Problem 1.12** Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|}$$

converges for each  $x \in \mathbb{R}$  and the sum is a Lipschitz function.

**Problem 1.13** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous and  $\{x_k\} \subset [a, b]$  a Cauchy sequence. Prove that  $f(x_k)$  is a Cauchy sequence.

**Problem 1.14** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be such that  $|f(x) - f(y)| \leq M|x - y|^\alpha$  for some  $M > 0$ ,  $\alpha > 1$  and for all  $x, y \in \mathbb{R}$ . Prove that  $f$  is constant.

**Problem 1.15** Let  $f(x) = x^2 \sin(1/x^2)$  for  $x \in [-1, 1]$ ,  $x \neq 0$ , and  $f(0) = 0$ . Show that  $f$  is differentiable on  $[-1, 1]$  but  $f'$  is unbounded on  $[-1, 1]$ .

**Problem 1.16** Suppose  $f_n \rightarrow f$  uniformly in  $\Omega$  open, and  $\{x_n\} \subset \Omega$  with  $x_n \rightarrow x \in \Omega$ . Prove that  $f_n(x_n) \rightarrow f(x)$ .

**Problem 1.17** If  $x_n$  is a sequence such that  $|x_{n+1} - x_n| \leq 1/2^n$ , then  $\{x_n\}$  is a Cauchy sequence.

**Problem 1.18** Let  $a < b, c < d$ . Define

$$f(x) = \begin{cases} ax \sin^2 \frac{1}{x} + bx \cos^2 \frac{1}{x}, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ cx \sin^2 \frac{1}{x} + dx \cos^2 \frac{1}{x}, & \text{for } x < 0. \end{cases}$$

Calculate  $D^-f, D^+f, D_-f, D_+f$  at  $x = 0$ .

**Problem 1.19** Let  $f$  be a continuous function in  $[-1, 2]$ . Given  $0 \leq x \leq 1$ , and  $n \geq 1$  define the sequence of functions

$$f_n(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t) dt.$$

Show that  $f_n$  is continuous in  $[0, 1]$  and  $f_n$  converges uniformly to  $f$  in  $[0, 1]$ .

**Problem 1.20** Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  be a uniformly bounded sequence of functions. Show that for each countable subset  $S \subset \mathbb{R}$  there exists a subsequence of  $f_n$  which converges in  $S$ .

Hint: select the subsequence by using a diagonal process

**Problem 1.21** Let  $f$  be a continuous function in  $[0, 1]$  such that  $f$  is absolutely continuous in  $[0, \epsilon]$  for every  $\epsilon, 0 < \epsilon < 1$ . Show that  $f$  is absolutely continuous in  $[0, 1]$ .

**Problem 1.22** Let  $f_n$  be absolutely continuous functions in  $[a, b]$ ,  $f_n(a) = 0$ . Suppose  $f'_n$  is a Cauchy sequence in  $L^1([a, b])$ . Show that there exists  $f$  absolutely continuous in  $[a, b]$  such that  $f_n \rightarrow f$  uniformly in  $[a, b]$ .

**Problem 1.23** Let  $f_n(x) = \cos(nx)$  on  $\mathbb{R}$ . Prove that there is no subsequence  $f_{n_k}$  converging uniformly in  $\mathbb{R}$ .

**Problem 1.24** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a bounded function. If  $E \subset \mathbb{R}^n$ , then the oscillation of  $f$  over  $E$  is defined by

$$\text{osc}_E f = \sup_E f - \inf_E f,$$

and for  $x \in \mathbb{R}^n$  define  $\omega_f(x) = \lim_{\delta \rightarrow 0} \text{osc}_{B_\delta(x)} f$ . Prove

1. If  $E \subset F$ , then  $\text{osc}_E f \leq \text{osc}_F f$ .
2. Prove that  $\omega_f$  is well defined and  $\omega_f(x) = \inf_{\delta > 0} \text{osc}_{B_\delta(x)} f$ .
3. For each  $\alpha > 0$  the set  $\{x : \omega_f(x) < \alpha\}$  is open.
4. The set of points of discontinuity of  $f$  is an  $F_\sigma$ .

## 2 Semicontinuous functions

**Definition 2.1** Let  $\Omega \subset \mathbb{R}^n$  be open. The function  $f : \Omega \rightarrow \overline{\mathbb{R}}$  is lower (upper) semicontinuous if  $f(z) \leq \liminf_{x \rightarrow z} f(x) = \lim_{\delta \rightarrow 0} \inf_{|x-z| < \delta} f(x)$  ( $f(z) \geq \limsup_{x \rightarrow z} f(x) = \lim_{\delta \rightarrow 0} \sup_{|x-z| < \delta} f(x)$ ).

**Problem 2.2** Prove that the following statements are equivalent:

1.  $f$  is lower semicontinuous;
2.  $f^{-1}(c, +\infty)$  is open for each  $c \in \mathbb{R}$ ;
3.  $f^{-1}(-\infty, c]$  is closed for each  $c \in \mathbb{R}$ ;
4. If  $x_k \rightarrow z$ , then  $f(z) \leq \sup_k f(x_k)$ ;
5. For each  $z \in \Omega$  and for each  $\epsilon > 0$  there exists a neighborhood  $V$  of  $z$  such that  $f(x) \geq f(z) - \epsilon$  for each  $x \in V$ .

**Problem 2.3** Let  $g : \Omega \rightarrow \mathbb{R}$  and define

$$f(x) = \liminf_{z \rightarrow x} g(z).$$

Prove that  $f$  is lower semicontinuous.

**Problem 2.4** Prove that

1. if  $E \subset \mathbb{R}^n$ , then  $\chi_E$  is lower semicontinuous if and only if  $E$  is open.
2. the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = m/n \text{ with } n \in \mathbb{N}, m \in \mathbb{Z}, \text{ and } m/n \text{ irreducible} \end{cases}$$

is upper semicontinuous.

**Problem 2.5** Let  $\{f_\alpha\}_{\alpha \in J}$  be a family of lower semicontinuous function in  $\Omega$ . Prove that  $f(x) = \sup_{\alpha \in J} f_\alpha(x)$  is lower semicontinuous.

**Problem 2.6** Prove that the function defined in Problem 1.1 is lower semicontinuous in  $\mathbb{R}$ .

### 3 Measure

**Problem 3.1** Let  $E_j$  be a sequence of sets in  $\mathbb{R}^n$  and  $E = \limsup_{j \rightarrow \infty} E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$ . Show that  $\limsup_{j \rightarrow \infty} \chi_{E_j}(x) = \chi_E(x)$ .

**Problem 3.2** Consider the set of rational numbers  $\mathbb{Q} = \{q_j\}_{j=1}^{\infty}$ . Prove that

$$\mathbb{R} \setminus \bigcup_{j=1}^{\infty} \left( q_j - \frac{1}{j^2}, q_j + \frac{1}{j^2} \right) \neq \emptyset.$$

**Problem 3.3** Construct a countable family of closed intervals contained in  $[0, 1]$  such that the union covers  $[0, 1]$  but there is no finite subcovering.

**Problem 3.4** Let  $E$  be a set in  $\mathbb{R}^n$ . Show that there exists a sequence  $G_i$  of open sets,  $G_1 \supset G_2 \supset \cdots \supset E$  such that

$$\left| \bigcap_{i=1}^{\infty} G_i \right| = |E|_e.$$

**Problem 3.5** If  $E \subset [0, 1]$ ,  $|E| = 0$  and  $f(x) = x^3$ , then  $|f(E)| = 0$ .

**Problem 3.6** True or false, justify your answer.

1. The class of Lebesgue measurable sets has cardinality  $2^c$ .
2. Every perfect set has positive measure.
3. Every bounded function is measurable.

**Problem 3.7** Let  $\mu^*$  be an exterior measure and let  $A_n$  be a sequence of sets such that

$$\sum_{n=1}^{\infty} \mu^*(A_n) < +\infty.$$

Show that  $\mu^*(\overline{\lim} A_n) = 0$ . ( $\overline{\lim} A_n = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j$ )

**Problem 3.8** Let  $f : [0, 1] \rightarrow [0, 1]$  be the Cantor function, and let  $\mu$  be the Lebesgue-Stieltjes measure generated by  $f$ , i.e.  $\mu((a, b]) = f(b) - f(a)$ , for  $(a, b] \subset [0, 1]$ . Show that  $\mu$  is singular with respect to Lebesgue measure.

**Problem 3.9** Let  $\Omega$  be an uncountable set. The class of sets defined by

$$F = \{A \subset \Omega : A \text{ is countable or } A^c \text{ is countable}\}$$

is a  $\sigma$ -algebra. Define the measures:  $\mu(A) = +\infty$  if  $A$  is infinite,  $\mu(A) = \#(A)$  if  $A$  is finite;  $\nu(A) = 0$  if  $A$  is countable and  $\nu(A) = 1$  if  $A$  is uncountable. Show that  $\nu \ll \mu$  but an integral representation of the form  $\nu(A) = \int_A f d\mu$  is not valid. Why this does not contradict the Radon-Nikodym theorem?

**Problem 3.10** Show that given  $\delta, 0 < \delta < 1$ , there exists a set  $E_\delta \subset [0, 1]$  which is perfect, nowhere dense and  $|E_\delta| = 1 - \delta$ .

Hint: the construction is similar to the construction of the Cantor set, except that at the  $k$ -th stage each interval removed has length  $\delta 3^{-k}$ .

**Problem 3.11** Let  $\{I_1, \dots, I_N\}$  be a finite family of intervals in  $\mathbb{R}$  such that  $\mathbb{Q} \cap [0, 1] \subset \cup_{j=1}^N I_j$ . Prove that  $\sum_{j=1}^N |I_j| \geq 1$ . Is this true if the family of intervals is infinite?

**Problem 3.12** Let  $\{E_j\}_{j=1}^\infty$  be a sequence of measurable sets in  $\mathbb{R}^n$  such that  $|E_j \cap E_i| = 0$  for  $j \neq i$ . Prove that

$$|\bigcup_{j=1}^\infty E_j| = \sum_{j=1}^\infty |E_j|.$$

**Problem 3.13** Let  $A, B \subset \mathbb{R}^n$  such that  $A$  is measurable. If  $A \cap B = \emptyset$  then  $|A \cup B|_e = |A| + |B|_e$ .

Hint: given  $\epsilon > 0$  there exists  $F$  closed such that  $F \subset A$  and  $|A \setminus F| < \epsilon$ .

**Problem 3.14** Let  $\{E_j\}_{j=1}^\infty$  be a sequence of measurable sets in  $\mathbb{R}^n$ . Prove that

1.  $\liminf_{j \rightarrow \infty} E_j$  and  $\limsup_{j \rightarrow \infty} E_j$  are measurable.
2.  $|\liminf_{j \rightarrow \infty} E_j| \leq \liminf_{j \rightarrow \infty} |E_j|$ .
3. If  $|\cup_{j \geq k} E_j| < \infty$  for some  $k$  then  $|\limsup_{j \rightarrow \infty} E_j| \geq \limsup_{j \rightarrow \infty} |E_j|$ .

**Problem 3.15** Let  $A \subset \mathbb{R}$  measurable with  $|A| < \infty$ . Show that the function  $f(x) = |(-\infty, x) \cap A|$  is nondecreasing, bounded and uniformly continuous in  $\mathbb{R}$ .

**Problem 3.16** Show that  $\mathbb{Q}$  is not a set of type  $G_\delta$ . Construct a set  $G$  of type  $G_\delta$  such that  $\mathbb{Q} \subset G$  and  $|G| = 0$ .

**Problem 3.17** Let  $E$  be a measurable set in  $\mathbb{R}$  with positive measure. We say that  $x \in \mathbb{R}$  is a point of positive measure with respect to  $E$  if  $|E \cap I| > 0$  for each open interval  $I$  containing  $x$ . Let  $E_+ = \{x \in \mathbb{R} : x \text{ is of positive measure with respect to } E\}$ . Prove that

1.  $E_+$  is perfect.
2.  $|E \setminus E_+| = 0$ .

**Problem 3.18** Let  $E$  be the set of numbers in  $[0, 1]$  whose binary expansion has 0 in all the even places. Show that  $|E| = 0$ .

**Problem 3.19** Let  $f_k$  be measurable and  $f_k \rightarrow f$  a.e. in  $\mathbb{R}^n$ . Prove that there exists a sequence of measurable sets  $\{E_j\}_{j=1}^\infty$  such that  $|\mathbb{R}^n \setminus \cup_{j=1}^\infty E_j| = 0$  and  $f_k \rightarrow f$  uniformly on each  $E_j$ .

**Problem 3.20** Let  $E \subset \mathbb{R}^n$  be a measurable set with positive measure, and let  $D \subset \mathbb{R}^n$  be a countable dense set. Prove that  $|\mathbb{R}^n \setminus \cup_{q_k \in D} (q_k + E)| = 0$ .

**Problem 3.21** An ellipsoid in  $\mathbb{R}^n$  with center at  $x_0$  is a set of the form

$$E = \{x \in \mathbb{R}^n : \langle A(x - x_0), x - x_0 \rangle \leq 1\},$$

where  $A$  is an  $n \times n$  positive definite symmetric matrix and  $\langle \cdot, \cdot \rangle$  denotes the Euclidean inner product. Prove that

$$|E| = \frac{\omega_n}{(\det A)^{1/2}},$$

where  $\omega_n$  is the volume of the unit ball.

Hint: formula of change of variables.

**Problem 3.22** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable function. Define  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$\Phi(x, y) = (x + f(x + y), y - f(x + y)).$$

Prove that  $|\Phi(E)| = |E|$  for each measurable set  $E \subset \mathbb{R}^2$ .

Hint: formula of change of variables.

**Problem 3.23** Let  $E \subset [0, 1]$  such that there exists  $\delta > 0$  satisfying  $|E \cap [a, b]| \geq \delta(b - a)$  for all  $[a, b] \subset [0, 1]$ . Prove that  $|E| = 1$ .

**Problem 3.24** We say that the sets  $A, B \subset \mathbb{R}^n$  are congruent if  $A = z + B$  for some  $z \in \mathbb{R}^n$ .

Let  $E \subset \mathbb{R}^n$  be measurable such that  $0 < |E| < +\infty$ . Suppose that there exists a sequence of disjoint sets  $\{E_i\}_{i=1}^{\infty}$  such that for all  $i, j$ ,  $E_i$  and  $E_j$  are congruent, and  $E = \cup_{j=1}^{\infty} E_j$ .

Prove that all the  $E_j$ 's are non-measurable.

## 4 Measurable functions

Let  $E$  be a measurable set and  $f_n : E \rightarrow \overline{\mathbb{R}}$  a sequence of measurable functions.

- (A)  $f_n \rightarrow f$  pointwise in  $E$  if  $f_n(x) \rightarrow f(x)$  for all  $x \in E$ .
- (B)  $f_n \rightarrow f$  pointwise almost everywhere in  $E$  if  $f_n(x) \rightarrow f(x)$  for almost all  $x \in E$ .
- (C)  $f_n \rightarrow f$  almost uniformly in  $E$  if for each  $\epsilon > 0$  there exists a measurable set  $F \subset E$  such that  $|E \setminus F| \leq \epsilon$  and  $f_n \rightarrow f$  uniformly in  $F$ .
- (D)  $f_n \rightarrow f$  in measure in  $E$  if for each  $\epsilon > 0$ ,  $\lim_{n \rightarrow \infty} |\{x \in E : |f_n(x) - f(x)| \geq \epsilon\}| = 0$ .
- (E)  $f_n$  is a Cauchy sequence in measure in  $E$  if for each  $\epsilon, \delta > 0$ , there exists  $N$  such that  $|\{x \in E : |f_n(x) - f_m(x)| \geq \epsilon\}| \leq \delta$  for all  $n, m \geq N$ .

$f : E \rightarrow \mathbb{R}$  is a measurable function if  $f^{-1}((-\infty, a))$  is a Lebesgue measurable set for all  $a \in \mathbb{R}$ .

$f : E \rightarrow \mathbb{R}$  is a Borel measurable function if  $f^{-1}((-\infty, a))$  is a Borel set for all  $a \in \mathbb{R}$ .

**Problem 4.1** If  $f_n \rightarrow f$  almost uniformly in  $E$ , then  $f_n \rightarrow f$  in measure in  $E$ .

**Problem 4.2** If  $f_n \rightarrow f$  in measure and  $f_n \rightarrow g$  in measure, then  $f = g$  a.e.

**Problem 4.3** If  $f_n$  is a Cauchy sequence in measure, then there exists a subsequence  $f_{n_k}$  that is a Cauchy sequence almost uniformly.

**Problem 4.4** If  $f_n$  is a Cauchy sequence in measure, then there exists  $f$  measurable such that  $f_n \rightarrow f$  in measure.

**Problem 4.5** Give an example of a sequence  $f_n$  that converges in measure but does not converge a.e.

**Problem 4.6** If  $|E| < \infty$  and  $f_n \rightarrow f$  a.e. in  $E$ , then  $f_n \rightarrow f$  in measure. Show that this is false if  $|E| = \infty$ .

**Problem 4.7** If  $f_n \rightarrow f$  in measure, then there exists a subsequence  $f_{n_k}$  such that  $f_{n_k} \rightarrow f$  a.e.

**Problem 4.8** If  $f_n \rightarrow f$  in measure, then  $f_n$  is a Cauchy sequence in measure.

**Problem 4.9** Let  $f, f_k : E \rightarrow \mathbb{R}$  be measurable functions. Prove that

1. If  $f_k \rightarrow f$  in measure, then any subsequence  $f_{k_j}$  contains a subsequence  $f_{k_{j_m}} \rightarrow f$  a.e. as  $m \rightarrow \infty$ .
2. Suppose  $|E| < \infty$ . If any subsequence  $f_{k_j}$  contains a subsequence  $f_{k_{j_m}} \rightarrow f$  a.e. as  $m \rightarrow \infty$ , then  $f_k \rightarrow f$  in measure.

Hint: for (2) suppose by contradiction that  $f_k \not\rightarrow f$  in measure. Then there exists  $\epsilon_0 > 0$  such that  $|\{x \in E : |f_k(x) - f(x)| \geq \epsilon_0\}| \not\rightarrow 0$ . Hence there is an increasing sequence  $k_j \rightarrow \infty$  such that  $|\{x \in E : |f_{k_j}(x) - f(x)| \geq \epsilon_0\}| \geq r > 0$ , for some  $r > 0$  and for all  $j$ . By hypothesis there is a subsequence  $f_{k_{j_m}} \rightarrow f$  a.e. as  $m \rightarrow \infty$ . Now use Egorov to get a contradiction.

**Problem 4.10** Let  $f : E \rightarrow \mathbb{R}$  be a measurable function. Prove that if  $B \subset \mathbb{R}$  is a Borel set then  $f^{-1}(B)$  is measurable.

Hint: consider  $\mathcal{A} = \{A \subset \mathbb{R} : f^{-1}(A) \text{ is measurable}\}$  and show that  $\mathcal{A}$  is a  $\sigma$ -algebra that contains the open sets of  $\mathbb{R}$ .

**Problem 4.11** If  $f : E \rightarrow \mathbb{R}$  is a measurable function and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is Borel measurable, then  $g \circ f$  is measurable.

## 5 Integration

**Problem 5.1** Let  $f : [a, b] \rightarrow \mathbb{R}$  be integrable and nonnegative. Prove that

$$\left( \int_a^b f(x) \cos x \, dx \right)^2 + \left( \int_a^b f(x) \sin x \, dx \right)^2 \leq \left( \int_a^b f(x) \, dx \right)^2.$$

Hint: write  $f(x) = \sqrt{f(x)} \sqrt{f(x)} \cos x$  and use Schwartz on the left hand side.

**Problem 5.2** Show that

$$\lim_{n \rightarrow \infty} \int_{[0,1]^n} \frac{x_1^2 + \cdots + x_n^2}{x_1 + \cdots + x_n} \, dx = 2/3.$$

Hint: see Makarov, prob. 6.21, p.107.

**Problem 5.3** Let  $f_1 : [0, M] \rightarrow \mathbb{R}$  be a bounded function. Define

$$f_{n+1}(x) = \int_0^x f_n(t) \, dt, \quad n = 1, 2, \dots.$$

Prove that the series  $\sum_{n=2}^{\infty} f_n(x)$  is uniformly convergent on  $[0, M]$  and the sum is a continuous function on  $[0, M]$ .

**Problem 5.4** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be integrable. Prove that

1.  $|\{x \in \mathbb{R}^n : |f(x)| = \infty\}| = 0$ .
2. For each  $\epsilon > 0$ ,  $|\{x \in \mathbb{R}^n : |f(x)| \geq \epsilon\}| < \infty$ .
3. For each  $\epsilon > 0$  there exists a compact set  $K$  such that  $\int_{\mathbb{R}^n \setminus K} |f(x)| \, dx < \epsilon$ .
4. For each  $\epsilon > 0$  there exist  $M \in \mathbb{R}$  and  $A \subset \mathbb{R}^n$  measurable such that  $|f(x)| \leq M$  in  $A$  and  $\int_{\mathbb{R}^n \setminus A} |f(x)| \, dx < \epsilon$ .
5. For each  $\epsilon > 0$  there exist  $\delta > 0$  such that if  $A \subset \mathbb{R}^n$  is measurable and such that  $|A| < \delta$  then  $\int_A |f(x)| \, dx < \epsilon$ .

**Problem 5.5** Let  $f : E \rightarrow \mathbb{R}$  be measurable with  $|E| < \infty$ .

Then  $f$  is integrable if and only if  $\sum_m |\{x \in \mathbb{R}^n : |f(x)| \geq m\}| < \infty$ .

Hint: use Abel's summation by parts formula,

$$\sum_{k=1}^N a_k b_k = A_N b_N + \sum_{k=1}^{N-1} A_k (b_k - b_{k+1}),$$

where  $A_k = \sum_{j=1}^k a_j$ .

**Problem 5.6** Let  $f, f_k : \mathbb{R}^n \rightarrow \mathbb{R}$  be integrable functions such that  $f_k \rightarrow f$  a.e. Then  $\int_{\mathbb{R}^n} |f_k(x) - f(x)| dx \rightarrow 0$  if and only if  $\int_{\mathbb{R}^n} |f_k(x)| dx \rightarrow \int_{\mathbb{R}^n} |f(x)| dx$ .

**Problem 5.7** Let  $P \subset [0, 1]$  be a perfect nowhere dense set, i.e.,  $(\overline{P})^\circ = \emptyset$ , with positive measure. Show that  $\chi_P(x)$  is not Riemann integrable on  $[0, 1]$  but it is Lebesgue integrable on  $[0, 1]$ .

**Problem 5.8** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be integrable and  $\epsilon > 0$ . Prove that

1. There exists  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  simple and integrable such that  $\int_{\mathbb{R}^n} |f(x) - g(x)| dx < \epsilon$ .
2. There exists  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  continuous with compact support such that  $\int_{\mathbb{R}^n} |f(x) - h(x)| dx < \epsilon$ .

**Problem 5.9 (Fatou in measure)** If  $f_k \geq 0$  and  $f_k \rightarrow f$  in measure in  $E$  then

$$\int_E f(x) dx \leq \liminf_{k \rightarrow \infty} \int_E f_k(x) dx.$$

Hint: let  $A_k = \int_E f_k(x) dx$  and  $A = \liminf_{k \rightarrow \infty} A_k$ ; there exists a subsequence  $A_{k_m} \rightarrow A$  as  $m \rightarrow \infty$ . Since  $f_{k_m} \rightarrow f$  in measure, there exists a subsequence  $f_{k_{m_j}} \rightarrow f$  a.e. as  $j \rightarrow \infty$ . Now apply Fatou.

**Problem 5.10 (Monotone convergence in measure)** If  $f_k \geq 0$ ,  $f_k \leq f_{k+1}$ , and  $f_k \rightarrow f$  in measure in  $E$  then

$$\int_E f(x) dx = \lim_{k \rightarrow \infty} \int_E f_k(x) dx.$$

**Problem 5.11 (Lebesgue in measure)** If  $|f_k| \leq g$  a.e. with  $g$  integrable in  $E$ , and  $f_k \rightarrow f$  in measure in  $E$  then

$$\int_E f(x) dx = \lim_{k \rightarrow \infty} \int_E f_k(x) dx.$$

**Problem 5.12** Let  $f, f_k : \mathbb{R}^n \rightarrow \mathbb{R}$  be measurable functions such that  $f_k \rightarrow f$  a.e. and there exists  $g$  integrable such that  $|f_k(x)| \leq g(x)$  a.e. for all  $k$ . Prove that  $f_k \rightarrow f$  almost uniformly.

**Problem 5.13** Let  $|E| < \infty$  and let  $X$  denote the class of measurable functions in  $E$ . If  $f, g \in X$  define

$$d(f, g) = \int_E \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} dx.$$

Show that  $(X, d)$  is a metric space and  $f_k \rightarrow f$  in  $(X, d)$  if and only if  $f_k \rightarrow f$  in measure in  $E$ .

**Problem 5.14** Let  $C \subset [0, 1]$  be the Cantor set and  $f : [0, 1] \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} 0, & \text{if } x \in C \\ p, & \text{if } x \in I \subset [0, 1] - P, I \text{ interval such that } \ell(I) = 3^{-p} \end{cases}$$

Show that  $f$  is measurable and  $\int_0^1 f(x) dx = 3$ .

**Problem 5.15** Prove that

1.  $\int_0^1 \frac{x \sin x}{1 + (nx)^a} dx = o\left(\frac{1}{n}\right)$ , if  $a > 1$ .
2.  $\lim_{n \rightarrow \infty} \int_0^\infty \frac{dx}{(1 + \frac{x}{n})^n x^{1/n}} = 1$

**Problem 5.16** Calculate  $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$  where:

1.  $f_n(x) = \frac{\log(x+n)}{n} e^{-x} \cos x$
2.  $f_n(x) = \frac{nx \log x}{1 + n^2 x^2}$
3.  $f_n(x) = \frac{n\sqrt{x}}{1 + n^2 x^2}$
4.  $f_n(x) = \frac{nx}{1 + n^2 x^2}$
5.  $f_n(x) = \frac{n^{3/2} x}{1 + n^2 x^2}$
6.  $f_n(x) = \frac{n^p x^r \log x}{1 + n^2 x^2}$ ,  $r > 0$ ,  $p < \min\{2, 1 + r\}$

7.  $f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)$  in the interval  $(0, \infty)$ .

8.  $f_n(x) = \frac{1 + nx}{(1 + x)^n}$

**Problem 5.17** Prove the following identities.

1.  $\lim_{n \rightarrow \infty} n^{1/\beta} \int_0^1 x^{1/\beta} (1-x)^n \frac{dx}{x} = \beta \int_0^\infty e^{-u^\beta} du$  if  $\beta > 0$ .

2.  $\lim_{n \rightarrow \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{(\alpha-1)} dx = \int_0^\infty e^{-x} x^{(\alpha-1)} dx$  if  $\alpha > 0$ .

3.  $\lim_{n \rightarrow \infty} \int_\alpha^\infty \sqrt{n} e^{-nx^2} dx = \int_\alpha^\infty \left(\lim_{n \rightarrow \infty} \sqrt{n} e^{-nx^2}\right) dx$  if  $\alpha > 0$ .

4.  $\int_0^1 \frac{x^{1/3}}{1-x} \log\left(\frac{1}{x}\right) dx = 9 \sum_{n=1}^\infty \frac{1}{(3n+4)^2}$ .

5.  $\int_0^\infty \frac{\sin t}{e^t - x} dt = \sum_{n=1}^\infty \frac{x^{n-1}}{n^2 + 1}$  for  $-1 < x < 1$ .

6.  $\int_0^1 \log x \sin x dx = \sum_{n=1}^\infty \frac{(-1)^n}{(2n)(2n)!}$ .

**Problem 5.18** If  $f$  is Riemann integrable on  $[a, b]$  and  $f(x) = 0$  for  $x \in [a, b] \cap \mathbb{Q}$ , then  $\int_a^b f(x) dx = 0$ .

**Problem 5.19** Show that

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable on  $[0, 1]$ . Find the value of  $\int_0^1 f(x) dx$ .

**Problem 5.20** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be Riemann integrable. Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(k/n) = \int_0^1 f(x) dx.$$

**Problem 5.21** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous, and  $f(x) \neq 0$  for each  $x \in (0, 1)$ . Suppose that

$$f(x)^2 = 2 \int_0^x f(t) dt$$

for each  $x \in [0, 1]$ . Prove that  $f(x) \equiv x$ .

**Problem 5.22** Let  $f$  be a measurable function on  $[0, 1]$  and let

$$A = \{x \in [0, 1] : f(x) \in \mathbb{Z}\}.$$

Prove that  $A$  is measurable and

$$\int_0^1 [\cos(\pi f(x))]^{2n} dx \rightarrow |A|,$$

as  $n \rightarrow \infty$ .

**Problem 5.23** Find all the values of  $p$  and  $q$  such that the integral

$$\int_{x^2+y^2 \leq 1} \frac{1}{x^{2p} + y^{2q}} dx dy$$

converges.

**Problem 5.24** Let  $f_1, \dots, f_k$  be continuous real valued functions on the interval  $[a, b]$ . Show that the set  $\{f_1, \dots, f_k\}$  is linearly dependent on  $[a, b]$  if and only if the  $k \times k$  matrix with entries

$$\langle f_i, f_j \rangle = \int_a^b f_i(x) f_j(x) dx$$

has determinant zero.

**Problem 5.25** Let  $f : [0, +\infty) \rightarrow \mathbb{R}$  be continuously differentiable with compact support in  $[0, +\infty)$ ; and  $0 < a < b < \infty$ . Prove that

$$\int_0^\infty \frac{f(bx) - f(ax)}{x} dx = -f(0) \ln(b/a).$$

**Problem 5.26** Find all the values of  $p$  such that the integral

$$\int_0^\infty \int_0^{\pi/2} e^{-xy^p} \sin x dx dy$$

converges.

**Problem 5.27** Recall that the convolution of two integrable functions  $f$  and  $g$  is defined by

$$f \star g(x) = \int_{\mathbb{R}^n} f(x-y)g(y) dy.$$

Let  $f \geq 0$  such that  $\int_{\mathbb{R}^n} f(x) dx = A < 1$ . Define the sequence  $f_k = f \star \cdots \star f$  where the convolution is performed  $k$  times.

Prove that all the  $f_k$ 's are integrable and  $f_k \rightarrow 0$  in  $L^1(\mathbb{R}^n)$ .

**Problem 5.28** Let  $f \in L^1(0, +\infty)$  be nonnegative. Prove that

$$\frac{1}{n} \int_0^n x f(x) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

**Problem 5.29** Let  $f \geq 0$  in  $R$ , set  $g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ . Show that if  $g \in L(R)$  then  $f = 0$  a.e.

**Problem 5.30** Given  $f \in L^2([0, 1])$  define

$$Kf(x) = \frac{1}{x^{4/3}} \int_0^x f(t) dt.$$

Show that  $\|Kf\|_1 \leq C\|f\|_2$  where  $C$  is a constant independent of  $f$ .

Hint: use Cauchy-Schwartz.

**Problem 5.31** Let  $A$  be a measurable subset of  $[0, 2\pi]$ , show that

$$\int_A \cos nx dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

**Problem 5.32** Let  $f$  be a non-negative measurable function in  $[0,1]$ . Show that the limit

$$\lim_{n \rightarrow \infty} \int_0^1 f(x)^n dx$$

exists if and only if

$$|\{x \in [0, 1] : f(x) > 1\}| = 0.$$

**Problem 5.33** Prove that  $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \frac{\sin x}{1 + nx^2} dx = 0$ .

**Problem 5.34 (Variant of Fatou)** Let  $g_n \in L^1(E)$  and  $g_n \rightarrow g$  in  $L^1(E)$ . Suppose  $f_n$  are measurable functions on  $E$  and  $f_n \leq g_n$  a.e. for each  $n$ . Prove that

$$\limsup_{n \rightarrow \infty} \int_E f_n \leq \int_E \limsup_{n \rightarrow \infty} f_n.$$

**Problem 5.35 (Variant of Lebesgue)** Let  $g_n \in L^1(E)$  and  $g_n \rightarrow g$  in  $L^1(E)$ . Suppose  $f_n$  are measurable functions on  $E$  such that  $f_n \rightarrow f$  a.e. or  $f_n \rightarrow f$  in measure, and  $|f_n| \leq g_n$  a.e. for each  $n$ . Prove that

$$\int_E |f_n - f| \rightarrow 0,$$

as  $n \rightarrow \infty$ , and consequently  $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$ .

**Problem 5.36** Let  $|E| < \infty$ . Prove that  $f_n \rightarrow f$  in measure if and only if

$$\int_E \{|f_n - f| \wedge 1\} \rightarrow 0$$

as  $n \rightarrow \infty$ .

**Problem 5.37** Let  $f \in L^1(0, 1)$  and suppose that  $\lim_{x \rightarrow 1^-} f(x) = A < \infty$ . Show that

$$\lim_{n \rightarrow \infty} n \int_0^1 x^n f(x) dx = A.$$

## 6 Solutions Continuity

1.1 Let  $y < x$  with  $y$  irrational. Write

$$f(x) - f(y) = \sum_{\{n: y \leq r_n < x\}} 1/2^n.$$

Given  $\epsilon > 0$  there exists  $N$  such that  $\sum_{n \geq N} 1/2^n < \epsilon$ . Let  $\delta = \min\{|y - r_k| : 1 \leq k \leq N\}$ . If  $x - y < \delta$  and  $n$  is such that  $y \leq r_n < x$ , then  $n > N$ . Because if  $n \leq N$  and  $y \leq r_n < x$ , then  $r_n - y < x - y < \delta$  which is impossible by definition of  $\delta$ .

If  $y \in \mathbb{Q}$ , then  $y = r_m$  for some  $m$  and so  $f(x) - f(y) = \sum_{\{n: y \leq r_n < x\}} 1/2^n \geq 1/2^m$  for any  $x > y$ .

We have

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \sum_{n=1}^{\infty} \chi_{A(x)}(n) \frac{1}{2^n} dx = \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 \chi_{A(x)}(n) dx \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 \chi_{(r_n, +\infty)}(x) dx = \sum_{n=1}^{\infty} \frac{1}{2^n} |[0, 1] \cap (r_n, +\infty)|. \end{aligned}$$

1.2 Write

$$\sin \sqrt{x + 4n^2 \pi^2} - \sin \sqrt{y + 4n^2 \pi^2} = \cos \sqrt{\xi + 4n^2 \pi^2} \frac{1}{2\sqrt{\xi + 4n^2 \pi^2}} (x - y),$$

with some  $\xi$  between  $x + 4n^2 \pi^2$  and  $y + 4n^2 \pi^2$ . To show the pointwise convergence pick  $y = 0$ . For (3),  $|f_n(z)| = 1$  when  $z + 4n^2 \pi^2 = (k + 1/2)\pi$ .

1.3 The class is bounded in  $C[a, b]$  and equicontinuous.

1.4 If  $|f(x)| \leq 1$  on  $[a, b]$ , then  $T(f)$  is bounded. Show that  $T(f)$  are equicontinuous writing

$$T(f)(x) - T(f)(z) = \int_a^b (-x^2 + z^2) f(y) dy + \int_a^b (e^{-x^2} - e^{-z^2}) e^y f(y) dy.$$

1.5 The continuity follows from the uniform convergence of the series. Given  $\epsilon > 0$  there exists  $N$  such that  $\sum_{k=N+1}^{\infty} 2^{-k} < \epsilon$ . Write

$$f_n(t) = \sum_{k=1}^N 2^{-k} g_n(t - r_k) + \sum_{k=N+1}^{\infty} 2^{-k} g_n(t - r_k) = A_n(t) + B_n(t).$$

We have  $\text{supp } g_n(\cdot - r_k) \subset [t - 1/2n, t]$ . Let  $E = \{r_1, r_2, \dots, r_N\}$ . Suppose first that  $t \notin E$  and let  $\delta = \text{dist}(t, E)$ . If  $1/2n < \delta$ , then  $r_k \notin [t - 1/2n, t]$  for all

$1 \leq k \leq N$  and so  $A_n(t) = 0$  for  $n > 1/2\delta$ . If on the other hand  $t \in E$ , then  $\limsup_{n \rightarrow \infty} A_n(t) \leq \sum_{k=1}^N 2^{-k} \limsup_{n \rightarrow \infty} g_n(t - r_k) = 0$ .

Let  $(a, b)$  be an interval. There exists  $r_m \in (a, b)$  and so  $r_m + 1/n \in (a, b)$  for all  $n$  sufficiently large. Therefore  $\sup_{x \in (a, b)} f_n(x) \geq f_n(r_m + 1/n) \geq 2^{-m} g_n(1/n) = 2^{-m}$  for all  $n$  sufficiently large.

**1.6**  $f_n(x)$  is a Riemann sum and converges to  $\int_x^{x+1} f(t) dt$  as  $n \rightarrow \infty$ . Let  $(a, b)$  be a finite interval. Write

$$f_n(x) - \int_x^{x+1} f(t) dt = \sum_{k=0}^{n-1} \int_{x+k/n}^{x+(k+1)/n} (f(x+k/n) - f(t)) dt.$$

Since  $f$  is uniformly continuous on  $[a, b+1]$  we have that  $|f(x+k/n) - f(t)| < \epsilon$  for all  $x \in [a, b]$ ,  $x+k/n \leq t \leq x+(k+1)/n$ ,  $0 \leq k \leq n-1$ , and for all  $n$  sufficiently large. Therefore

$$\sup_{x \in [a, b]} \left| f_n(x) - \int_x^{x+1} f(t) dt \right| < \epsilon$$

for all  $n$  sufficiently large.

**1.7** (a)  $\not\Rightarrow$  (b)  $f(x) = 1/x$ ; (b)  $\not\Rightarrow$  (a)  $g(x) = 1$ ,  $f(x) = 1$  for  $x \in \mathbb{Q}$  and  $f(x) = 0$  for  $x \in \mathbb{R} \setminus \mathbb{Q}$ .

**1.8** By the absolute continuity  $f(x) - f(a) = \int_a^x f'(t) dt$  and  $g(x) - g(a) = \int_a^x g'(t) dt$ , for all  $a \leq x \leq b$ , so  $f(x) - g(x) = f(a) - g(a)$  for all  $a \leq x \leq b$ .

**1.9**  $f(x)^2 - f(y)^2 = (f(x) + f(y))(f(x) - f(y))$  and  $f$  is bounded.

**1.10** Let  $h > 0$ . We have  $|g(x+h) - g(x)| \leq |g(x+h) - g(0)| + |g(x) - g(0)| \leq (x+h)^a + x^a \leq 2(x+h)^a$ . Also  $|g'(x)| \leq 2a/x$  for  $x \neq 0$ . By the mean value theorem,  $|g(x+h) - g(x)| \leq C_a \min\{(x+h)^a, h/x\}$ . Next either  $x^{a+1} \geq h$  or  $x^{a+1} < h$ , and then conclude  $g \in C^\alpha([0, 1])$ . To show the negative statement notice that  $g(x) = 0$  at  $x = \xi = (n\pi)^{-1/a}$  with  $n = 1, 2, \dots$ . Let  $\delta > 0$  small, and  $\xi' = ((n+\delta)\pi)^{-1/a}$ . Suppose by contradiction that  $|g(\xi) - g(\xi')| \leq K |\xi - \xi'|^\beta$ . We have  $|g'(x)| = \frac{a}{x} |x^a \sin x^{-a} - \cos x^{-a}| \geq \frac{a}{x} (|\cos x^{-a}| - x^a) \geq \frac{a}{x} (|\cos x^{-a}| - x^a)$ . From the mean value theorem  $|g(\xi) - g(\xi')| = |g(\bar{\xi})| |\xi - \xi'|$ , with  $((n+1)\delta)^{-1/a} \leq \bar{\xi} \leq (n\delta)^{-1/a}$ . Choosing  $\delta > 0$  small, we get that  $|g'(\bar{\xi})| \geq C_\delta n^{1/a}$  for all  $n$  large. Also notice that  $|\xi - \xi'| \approx n^{-(1+a)/a}$  for  $n$  large. Combining the orders of magnitude we obtain a contradiction unless  $\beta \leq a/(1+a)$ .

**1.11** Let  $\{f_n\}$  be a Cauchy sequence. Given  $x \in (a, b)$  take the partition  $\{a, x, b\}$ . We have

$$|f_n(x) - f_m(x)| \leq |(f_n - f_m)(x) - (f_n - f_m)(a)| + |(f_n - f_m)(a)| \leq \|f_n - f_m\|_{BV} \rightarrow 0,$$

as  $n, m \rightarrow \infty$ . Thus, for each  $x \in [a, b]$ ,  $\{f_n(x)\}$  is a Cauchy sequence. Let  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . Let  $\Gamma = \{x_0, \dots, x_N\}$  be a partition of  $[a, b]$ . Given  $\epsilon > 0$  there exists

$n_0$  such that

$$\sum_{i=1}^N |(f_n - f_m)(x_i) - (f_n - f_m)(x_{i-1})| < \epsilon$$

for each  $n, m \geq n_0$ . Letting  $n \rightarrow \infty$  yields

$$\sum_{i=1}^N |(f - f_m)(x_i) - (f - f_m)(x_{i-1})| < \epsilon$$

for each  $m \geq n_0$ . Hence  $\|f - f_m\|_{BV} < \epsilon$  for  $m \geq n_0$ .

Let  $f_n$  be absolutely continuous such that  $\|f_n - f\|_{BV} \rightarrow 0$ . Given  $\epsilon > 0$  there exists  $N$  such that  $\sum_{i=1}^k |(f_n - f)(x_i) - (f_n - f)(x_{i-1})| < \epsilon$  for each  $n \geq N$  and any partition  $\Gamma = \{x_0, \dots, x_k\}$ . Since  $f_N$  is absolutely continuous, there exists  $\delta > 0$  such that if  $(a_1, b_1), \dots, (a_r, b_r)$  is any collection of disjoint intervals of  $[a, b]$  with  $\sum_{i=1}^r (b_i - a_i) < \delta$ , then  $\sum_{i=1}^r |f_N(b_i) - f_N(a_i)| < \epsilon$ . If  $\Gamma = \{a, a_1, b_1, a_2, b_2, \dots, a_r, b_r, b\}$ , then

$$\begin{aligned} \sum_{i=1}^r |f(b_i) - f(a_i)| &\leq \sum_{i=1}^k |(f - f_N)(b_i) - (f - f_N)(a_i)| + \sum_{i=1}^k |f_N(b_i) - f_N(a_i)| \\ &\leq \|f - f_N\|_{BV} + \epsilon < 2\epsilon. \end{aligned}$$

## 7 Solutions Semicontinuity

**2.6** Suppose  $y \in \mathbb{Q}$ ,  $y = r_m$ . Then  $f(x) \geq \frac{1}{2^m} + f(y)$  for all  $x > y$ , so

$$\inf_{y < x < y + \delta} f(x) \geq \frac{1}{2^m} + f(y) > f(y)$$

for any  $\delta > 0$ . We now estimate  $\inf_{y - \delta < x \leq y} f(x)$ . If  $x \leq y$ , then

$$f(x) = f(y) - \sum_{\{n: x \leq r_n < y\}} 1/2^n.$$

Suppose  $y - \delta < x \leq y$ , then  $\{n : x \leq r_n < y\} \subset \{n : y - \delta < r_n < y\}$  and so

$$f(x) \geq f(y) - \sum_{\{n: y - \delta < r_n < y\}} 1/2^n.$$

We claim that for each  $N \geq 1$  there exists  $\delta > 0$  such that if  $r_j \in (r_m - \delta, r_m)$  then  $j \geq N$ . Suppose by contradiction that this is false. Then there exists  $N_0 \geq 1$  such that for each  $\delta = 1/k$  there exists a  $j(k)$  such that  $r_{j(k)} \in (r_m - 1/k, r_m)$  and  $j(k) < N_0$ . Hence  $j(k)$  takes only a finite number of values between 1 and  $N_0$ , and therefore there exists a subsequence  $k_i$  such that  $k_i \rightarrow \infty$  as  $i \rightarrow \infty$  with  $j(k_i) = s$  and  $r_s \in (r_m - 1/k_i, r_m)$  for all  $i$ . Letting  $i \rightarrow \infty$  we get  $r_m \leq r_s < r_m$ , a contradiction.

Now given  $\epsilon > 0$ , there exists  $N \geq 1$  such that  $\sum_{n \geq N} 1/2^n < \epsilon$ , and by the claim there exists  $\delta_0 > 0$  such that  $\{n : y - \delta_0 < r_n < y\} \subset \{n : n \geq N\}$ . Consequently,

$$\sum_{\{n: y - \delta_0 < r_n < y\}} 1/2^n \leq \sum_{n \geq N} 1/2^n < \epsilon,$$

and so  $f(x) \geq f(y) - \epsilon$  for  $y - \delta < x \leq y$  and  $0 < \delta \leq \delta_0$ . Thus,

$$\inf_{y - \delta < x \leq y} f(x) \geq f(y) - \epsilon$$

for all  $0 < \delta \leq \delta_0$ , and so  $\lim_{\delta \rightarrow 0} \inf_{y - \delta < x \leq y} f(x) \geq f(y) - \epsilon$  for any  $\epsilon > 0$ .