Real Analysis I, Prof. Gutiérrez
Problems on Lebesgue measure
Week of September 21, 2009

Notation.

- If $E \subset \mathbb{R}^n$ the outer measure of E is given by $|E|_e = \inf\{|G| : E \subset G \text{ with } G \text{ open}\}$.
- If a_k is a sequence of real numbers then

$$\limsup_{k\to\infty} a_k = \lim_{j\to\infty} \sup\{a_k : k \geq j\}, \qquad \liminf_{k\to\infty} a_k = \lim_{j\to\infty} \inf\{a_k : k \geq j\}.$$

- If E_j is a sequence of sets then

$$\liminf_{j\to\infty} E_j = \cup_{j=1}^{\infty} \cap_{m=j}^{\infty} E_m, \qquad \limsup_{j\to\infty} E_j = \cap_{j=1}^{\infty} \cup_{m=j}^{\infty} E_m.$$

1. Let $f : \Omega \to R$ be a bounded function with $\Omega \subset \mathbb{R}^n$ a bounded open set. Define the oscillation of f at the point x by

$$\operatorname{osc}(f, x) = \lim_{h\to 0^+} \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in B_h(x) \cap \Omega\}.$$

   Prove that

   (a) $\operatorname{osc}(f, x) = \inf_{h>0} \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in B_h(x) \cap \Omega\}$.

   (b) If $g : \Omega \to \mathbb{R}$ is bounded, then $\operatorname{osc}(f + g, x) \leq \operatorname{osc}(f, x) + \operatorname{osc}(g, x)$.

   (c) The set $\{x : \operatorname{osc}(f, x) < \alpha\}$ is open for all α.

   (d) f is continuous at x if and only if $\operatorname{osc}(f, x) = 0$.

   (e) The set $D(f)$ of points where f is discontinuous satisfies $D(f) = \cup_{k=1}^{\infty}\{x : \operatorname{osc}(f, x) \geq 1/k\}$. And therefore the set of points of discontinuity of f is a set of type F_σ.

2. Recall the Baire category theorem: *If (X, d) is a complete metric space, then X is of the second category; that is, X cannot be written as a countable union of nowhere dense sets (a set E is nowhere dense if the interior of $\bar{E}$ is empty).* A set E in a metric space is perfect if it is closed and every point in E is a limit point of E (x is a limit point of a set E if for each $\epsilon > 0$ $B_\epsilon(x) \cap (E \setminus \{x\}) \neq \emptyset$, where $B_\epsilon(x)$ denotes the ball, in the metric d, with center x and radius ϵ).

   Prove

   (a) In $\mathbb{R}^n$ closed balls with positive radius are perfect sets.

   (b) If Y is a closed subset of the complete metric space (X, d), then Y is complete.

   (c) If E is perfect subset of the complete metric space (X, d), then E is uncountable.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that the graph of f, $G(f) = \{(x, f(x)) : x \in [a, b]\}$, is a set of measure zero in $\mathbb{R}^2$.

4. Let $\{I_1, \cdots, I_N\}$ be a finite family of open intervals in $\mathbb{R}$ such that $\mathbf{Q} \cap [0, 1] \subset \cup_{j=1}^{N} I_j$. Prove that $\sum_{j=1}^{N} |I_j| \geq 1$. Is this true if the family of intervals is infinite?

5. Let $0 < \delta < 1$. Following the construction of the Cantor set construct a subset of $[0, 1]$ that at the k-th stage the intervals removed have length $\delta/3^k$. Show that the resulting set is perfect, has measure $1 - \delta$ and is nowhere dense.

6. Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of measurable sets in $\mathbb{R}^n$. Prove that

   (a) $\liminf_{j\to\infty} E_j$ and $\limsup_{j\to\infty} E_j$ are measurable.

   (b) $|\liminf_{j\to\infty} E_j| \le \liminf_{j\to\infty} |E_j|$.

   (c) If $|\cup_{j\ge k} E_j| < \infty$ for some k then $|\limsup_{j\to\infty} E_j| \ge \limsup_{j\to\infty} |E_j|$.

7. Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of measurable sets in $\mathbb{R}^n$ such that $|E_j \cap E_i| = 0$ for $j \neq i$. Prove that

$$\left|\bigcup_{j=1}^{\infty} E_j\right| = \sum_{j=1}^{\infty} |E_j|.$$

8. Let $A, B \subset \mathbb{R}^n$ such that A is measurable. If $A \cap B = \emptyset$ then $|A \cup B|_e = |A| + |B|_e$.

   Hint: given $\epsilon > 0$ there exists F closed such that $F \subset A$ and $|A \setminus F| < \epsilon$. Also can be done using Carathéodory's condition.

9. Let $A \subset \mathbb{R}$ measurable with $|A| < \infty$. Show that the function $f(x) = |(-\infty, x) \cap A|$ is nondecreasing, bounded and uniformly continuous in $\mathbb{R}$.

10. Show that $\mathbf{Q}$ is not a set of type G_δ. Construct a set G of type G_δ such that $\mathbf{Q} \subset G$ and $|G| = 0$.

    Hint: for the first part use Baire category theorem.

11. Let E be a measurable set in $\mathbb{R}$ with positive measure. We say that $x \in \mathbb{R}$ is a point of positive measure with respect to E if $|E \cap I| > 0$ for each open interval I containing x. Let $E_+ = \{x \in \mathbb{R} : x \text{ is of positive measure with respect to } E\}$. Prove that

    1. E_+ is perfect.
    2. $|E \setminus E_+| = 0$.

12. Let E be the set of numbers in $[0, 1]$ whose binary expansion has 0 in all the even places. Show that $|E| = 0$.