

**Real Analysis I, Prof. Gutiérrez**  
**Problems on measurable functions**  
**Week of October 21, 2009**

**Notation.** Let  $E$  be a measurable set and  $f_n : E \rightarrow \mathbb{R}$  a sequence of measurable functions.

- $f_n \rightarrow f$  pointwise in  $E$  if  $f_n(x) \rightarrow f(x)$  for all  $x \in E$ .
- $f_n \rightarrow f$  pointwise almost everywhere in  $E$  if  $f_n(x) \rightarrow f(x)$  for almost all  $x \in E$ .
- $f_n \rightarrow f$  almost uniformly in  $E$  if for each  $\epsilon > 0$  there exists a measurable set  $F \subset E$  such that  $|E \setminus F| \leq \epsilon$  and  $f_n \rightarrow f$  uniformly in  $F$ .
- $f_n \rightarrow f$  in measure in  $E$  if for each  $\epsilon > 0$ ,  $\lim_{n \rightarrow \infty} |\{x \in E : |f_n(x) - f(x)| \geq \epsilon\}| = 0$ .
- $f_n$  is a Cauchy sequence in measure in  $E$  if for each  $\epsilon, \delta > 0$ , there exists  $N$  such that  $|\{x \in E : |f_n(x) - f_m(x)| \geq \epsilon\}| \leq \delta$  for all  $n, m \geq N$ .
- $f : E \rightarrow \mathbb{R}$  is a measurable function if  $f^{-1}((-\infty, a))$  is a Lebesgue measurable set for all  $a \in \mathbb{R}$ .
- $f : E \rightarrow \mathbb{R}$  is a Borel measurable function if  $f^{-1}((-\infty, a))$  is a Borel set for all  $a \in \mathbb{R}$ .

1. If  $f_n \rightarrow f$  in measure and  $f_n \rightarrow g$  in measure then  $f = g$  a.e.
2. If  $f_n \rightarrow f$  almost uniformly in  $E$  then  $f_n \rightarrow f$  in measure in  $E$ .
3. If  $f_n$  is a Cauchy sequence in measure then there exists  $f$  measurable such that  $f_n \rightarrow f$  in measure.
4. Give an example of a sequence  $f_n$  that converges in measure but does not converge a.e.
5. If  $|E| < \infty$  and  $f_n \rightarrow f$  a.e. in  $E$  then  $f_n \rightarrow f$  in measure. Show that this is false if  $|E| = \infty$ .
6. If  $f_n \rightarrow f$  in measure then  $f_n$  is a Cauchy sequence in measure.
7. Let  $f : E \rightarrow \mathbb{R}$  be a measurable function. Prove that if  $B \subset \mathbb{R}$  is a Borel set then  $f^{-1}(B)$  is measurable.

Hint: consider  $\mathcal{A} = \{A \subset \mathbb{R} : f^{-1}(A) \text{ is measurable}\}$  and show that  $\mathcal{A}$  is a  $\sigma$ -algebra that contains the open sets of  $\mathbb{R}$ .

8. If  $f : E \rightarrow \mathbb{R}$  is a measurable function and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is Borel measurable then  $g \circ f$  is measurable.
9. Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f_n(x) = \frac{1 - |x|^n}{1 + |x|^n}$ . Prove that
  - (a)  $f_n$  converges pointwise for each  $x \in \mathbb{R}$ .
  - (b)  $f_n$  does not converge uniformly in  $\mathbb{R}$ .
  - (c)  $f_n$  converges in measure in  $\mathbb{R}$ .
10. Determine whether the following sequences  $\{f_n\}_{n \geq 1}$  converge in measure and almost everywhere on the intervals  $I$ :

1.  $f_n(x) = \frac{nx}{1 + n^2x^2}; I = (0, 1)$ .

2.  $f_n(x) = \frac{nx}{1 + n^2x^2}; I = (0, +\infty)$ .

3.  $f_n(x) = nxe^{-nx^2}; I = (0, 1)$ .

4.  $f_n(x) = n^2 x e^{-nx^2}; I = (1, +\infty)$ .

5.  $f_n(x) = \frac{n^2 x^2}{n^4 + x^2}; I = (1, +\infty)$ .

11. Suppose  $f_k$  is a monotone function on  $(0, 1)$  for each  $k$  and  $f_k$  converges in measure to  $f$ . Prove that  $f$  is equal almost everywhere to a monotone function.

12. Suppose  $E$  has finite measure. Prove that  $f_n$  converges to  $f$  in measure in  $E$  if and only if each subsequence  $f_{n_j}$  contains a subsequence  $f_{n_{j_k}}$  converging to  $f$  as  $k \rightarrow \infty$  a.e. in  $E$ .

HINT: suppose  $f_n$  does not converge to  $f$  in measure then there exists  $\epsilon_0 > 0$  such that  $|\{x \in E : |f_n(x) - f(x)| > \epsilon_0\}| \not\rightarrow 0$  as  $n \rightarrow \infty$ . Show there exists an increasing sequence  $n_j \rightarrow \infty$  such that  $|\{x \in E : |f_{n_j}(x) - f(x)| > \epsilon_0\}| > r > 0$  for some  $r$  and for all  $j$ . Suppose  $f_{n_j}$  contains a subsequence  $f_{n_{j_k}}$  converging to  $f$  a.e. in  $E$ . Use Egorov to get a contradiction.