Real Analysis, Math 8041 (old 557), Prof. Gutiérrez
Problems on Bounded Variation and Riemann-Stieltjes Integration
Week of September 1, 2009

Notation.

- Given $f : [a, b] \to R$, we let

$$V_a^b f = \sup\{\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| : \{x_0, x_1, \cdots, x_m\} \text{ is a partition of } [a, b]\}.$$

We say $f \in BV[a, b]$ if $V_a^b f < \infty$.

- $\limsup_{k\to\infty} a_k = \lim_{j\to\infty} \sup\{a_k : k \geq j\}$.
- $a \vee b = \max\{a, b\}$; $a \wedge b = \min\{a, b\}$.
- $C[a, b]$ denotes the class of continuous functions on $[a, b]$.

1. If $f \in BV[a, b]$ then $|f| \in BV[a, b]$ and $V_a^b|f| \leq V_a^b f$. Show that the converse is false.

2. If $f, g \in BV[a, b]$ then $V_a^b(f + g) \leq V_a^b f + V_a^b g$.

3. If $f, g \in BV[a, b]$ then $f \vee g, f \wedge g \in BV[a, b]$.
   Hint: $a \vee b = \frac{1}{2}(a + b + |a - b|)$; $a \wedge b = \frac{1}{2}(a + b - |a - b|)$.

4. If $f \in C[a, b]$ and $g(x) = \int_a^x f(t)\,dt$ then $g \in BV[a, b]$.

5. If $f_n \to f$ point-wise in $[a, b]$ then $V_a^b f \leq \liminf_{n\to\infty} V_a^b f_n$.

6. If $f \in BV[a, b]$ then $f(x^+)$ and $f(x^-)$ exist for each $x \in (a, b)$. Also $f(a^+)$ and $f(b^-)$ exist.

7. Show an example of a function $f \in BV[0, 1]$ such that f' exists on $(0, 1)$ but is unbounded.

8. The mere existence of f' on $[a, b]$ does not imply that $f \in BV[a, b]$. Show that the function $f(x) = x^2 \sin(1/x^2)$ for $0 < x \leq 1$ and $f(0) = 0$ has derivative for each $x \in [0, 1]$ but is not of bounded variation.

9. If $f \in BV[a, b]$, then prove that $|f(x)| \leq |f(a)| + V_a^b f$ for all $x \in [a, b]$.

10. If $f, g \in BV[a, b]$ then $fg \in BV[a, b]$.

11. Find the following variations:

    1. $V_0^\pi f$ for $f(x) = \sin^2 x$.
    2. $V_0^2 f$ for $f(x) = x^3 - 3x + 4$.
    3. $V_0^a f$ for $f(x) = x^n e^{-x}$ where $0 < n < a$.

12. Prove that $f \in BV[0, 1]$ where $f(x) = x^2 \sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$.

13. Let $0 < \alpha \leq 1$. A function $f \in C^\alpha([0, 1])$, *f is Hölder continuous of order α*, if there exists $K \geq 0$ such that $|f(x) - f(y)| \leq K\,|x - y|^\alpha$ for all $x, y \in [0, 1]$. Prove that

    1. $C^\alpha[0, 1] \subset C^\beta([0, 1])$ for $0 < \beta \leq \alpha \leq 1$.

2. If f is Hölder continuous of order one, that is, f is Lipschitz, then $f \in BV[0, 1]$.
3. Let $a > 0$. Define $g(x) = x^a \sin x^{-a}$ for $x \neq 0$ and $f(0) = 0$.
4. Prove that $g \notin BV[0, 1]$. Prove that $g \in C^\alpha([0, 1])$ with $\alpha = a/(1 + a)$, and $g \notin C^\beta([0, 1])$ for $\beta > a/(1 + a)$.

HINT: Let $h > 0$. We have $|g(x+h)-g(x)| \leq |g(x+h)-g(0)|+|g(x)-g(0)| \leq (x+h)^a+x^a \leq 2(x+h)^a$. Also $|g'(x)| \leq 2a/x$ for $x \neq 0$. By the mean value theorem, $|g(x+h)-g(x)| \leq C_a \min\{(x+h)^a, h/x\}$. Next either $x^{a+1} \geq h$ or $x^{a+1} < h$, and then conclude $g \in C^\alpha([0, 1])$. To show the negative statement notice that $g(x) = 0$ at $x = \xi = (n\pi)^{-1/a}$ with $n = 1, 2, \cdots$. Let $\delta > 0$ small, and $\xi' = ((n + \delta)\pi)^{-1/a}$. Suppose by contradiction that $|g(\xi) - g(\xi')| \leq K |\xi - \xi'|^\beta$. We have $|g'(x)| = \dfrac{a}{x} |x^a \sin x^{-a} - \cos x^{-a}| \geq \dfrac{a}{x} (|\cos x^{-a}| - x^a |\sin x^{-a}|) \geq \dfrac{a}{x} (|\cos x^{-a}| - x^a)$. From the mean value theorem $|g(\xi) - g(\xi')| = |g(\bar{\xi})| |\xi - \xi'|$, with $((n + 1)\delta)^{-1/a} \leq \bar{\xi} \leq (n\delta)^{-1/a}$. Choosing $\delta > 0$ small, we get that $|g'(\bar{\xi})| \geq C_\delta n^{1/a}$ for all n large. Also notice that $|\xi - \xi'| \approx n^{-(1+a)/a}$ for n large. Combining the orders of magnitude we obtain a contradiction unless $\beta \leq a/(1 + a)$.

14. Consider the class $BV[a, b]$ of bounded variation functions in $[a, b]$ with the norm

$$\|f\|_{BV} = \sup_\Gamma \sum_{i=1}^{k} |f(\alpha_i) - f(\alpha_{i-1})| + |f(a)|,$$

where the supremum is taken over all partitions $\Gamma = \{\alpha_0, \ldots, \alpha_k\}$ of $[a, b]$.

Prove that $\| \cdot \|$ is a norm and $(BV[a, b], \| \cdot \|)$ is a Banach space.

15. Show that the Riemann-Stieltjes integral $\int_{-1}^{1} x\, d(e^{|x|})$ exists and calculate its value.

16. Let f be bounded on $[a, b]$ and ϕ nondecreasing on $[a, b]$. If $\int_a^b f(x)\, d(\phi(x))$ exists then $\int_a^b |f(x)|\, d(\phi(x))$ also exists and

$$\left| \int_a^b f(x)\, d(\phi(x)) \right| \leq \int_a^b |f(x)|\, d(\phi(x)).$$

Show that the converse is not true; there exists f bounded on $[0, 1]$ not Riemann integrable such that $|f|$ is Riemann integrable.

HINT: prove first that for f bounded, if we set $\bar{M} = \max_E |f|, \bar{m} = \min_E |f|, M = \max_E f, m = \min_E f$, then $\bar{M} - \bar{m} \leq M - m$. Conclude that $0 \leq \bar{U}_\Gamma - \bar{L}_\Gamma \leq U_\Gamma - L_\Gamma$, where U_Γ, L_Γ are the lower and upper sums for f and the partition Γ and similarly $\bar{U}_\Gamma$, $\bar{L}_\Gamma$ for $|f|$.

17. Let $f_n \to f$ uniformly in $[a, b]$ and $\phi_n \to \phi$ point-wise in $[a, b]$. Suppose that f_n are continuous on $[a, b]$ and $\phi_n \in BV[a, b]$ with $V_a^b \phi_n \leq M$ for all n. Prove that $\int_a^b f_n\, d\phi_n$ and $\int_a^b f\, d\phi$ exist and

$$\lim_{n\to\infty} \int_a^b f_n\, d\phi_n = \int_a^b f\, d\phi.$$

Let $f_n(x) = \dfrac{nx}{1+n^3x^2}$ and $\phi_n(x) = x^n$ for $0 \le x \le 1$. Show that $\lim_{n\to\infty} \int_a^b f_n \, d\phi_n$ exists and calculate its value.

HINT: write $\int_a^b f_n \, d\phi_n - \int_a^b f \, d\phi = \int_a^b (f_n - f) \, d\phi_n + \int_a^b f \, d(\phi_n - \phi)$. The second integral equals $\sum_{i=1}^k \int_{x_{i-1}}^{x_i} (f(x) - f(x_i)) \, d(\phi_n - \phi) + \sum_{i=1}^k \int_{x_{i-1}}^{x_i} f(x_i) \, d(\phi_n - \phi)$ for any fix partition $\{x_0, x_1, \cdots, x_k\}$.

18. If $f \in BV[a,b]$ then $\int_a^b df = f(b) - f(a)$.