The following lemma implies that the notion of generalized solution is closed under uniform limits. That is, if u_k are generalized solutions to $\det D^2 u = \nu$ in Ω and $u_k \to u$ uniformly on compact subsets of Ω, then u is also a generalized solution to $\det D^2 u = \nu$ in Ω.

Lemma 1.2.2 *Let $u_k \in C(\Omega)$ be convex functions such that $u_k \to u$ uniformly on compact subsets of Ω.*

Then:

(i) If $K \subset \Omega$ is compact, then

$$\limsup_{k\to\infty} \partial u_k(K) \subset \partial u(K),$$

and by Fatou

$$\limsup_{k\to\infty} |\partial u_k(K)| \leq |\partial u(K)|.$$

(ii) If K is compact and U is open such that $K \subset U \subset \overline{U} \subset \Omega$, then

$$\partial u(K) \subset \liminf_{k\to\infty} \partial u_k(U),$$

where the inequality holds for almost every point of the set on the left-hand side, and by Fatou

$$|\partial u(K)| \leq \liminf_{k\to\infty} |\partial u_k(U)|.$$

Proof. (i) If $p \in \limsup_{k\to\infty} \partial u_k(K)$, then for each n there exist k_n and $x_{k_n} \in K$ such that $p \in \partial u_{k_n}(x_{k_n})$. By selecting a subsequence x_j of x_{k_n} we may assume that $x_j \to x_0 \in K$. On the other hand,

$$u_j(x) \geq u_j(x_j) + p \cdot (x - x_j), \qquad \forall x \in \Omega,$$

and by letting $j \to \infty$, by the uniform convergence of u_j on compacts we get

$$u(x) \geq u(x_0) + p \cdot (x - x_0), \qquad \forall x \in \Omega,$$

that is $p \in \partial u(x_0)$.

(ii) Let $S = \{p : p \in \partial u(x_1) \cap \partial u(x_2) \quad \text{for some } x_1, x_2 \in \Omega,\, x_1 \neq x_2\}$. By Lemma 1.1.12, $|S| = 0$. Let $K \subset \Omega$ be compact and consider $\partial u(K) \setminus S$. If $p \in \partial u(K) \setminus S$, then there exists a unique $x_0 \in K$ such that $p \in \partial u(x_0)$ and $p \notin \partial u(x_1)$ for all $x_1 \in \Omega$, $x_1 \neq x_0$. Let U be open satisfying the assumptions. If $x_1 \in \Omega$ and $x_1 \neq x_0$, then $u(x_1) > u(x_0) + p \cdot (x_1 - x_0)$. Otherwise, $u(x_1) = u(x_0) + p \cdot (x_1 - x_0)$ and since $p \in \partial u(x_0)$ we have

$$\begin{aligned} u(x) &\geq u(x_0) + p \cdot (x - x_0) \qquad \forall x \in \Omega \\ &= u(x_1) - p \cdot (x_1 - x_0) + p \cdot (x - x_0) \\ &= u(x_1) + p \cdot (x - x_1) \qquad \forall x \in \Omega, \end{aligned}$$

that is, $p \in \partial u(x_1)$ which is impossible because we removed S from $\partial u(K)$. We may assume $\overline{U}$ is compact. Let $\ell(x) = u(x_0) + p \cdot (x - x_0)$ and set $\delta = \min\{u(x) - \ell(x) : x \in \partial U\} > 0$. We have that $|u(x) - u_k(x)| < \delta/2$ for all $x \in \overline{U}$ and for all $k \geq k_0$. Now let

$$\delta_k = \max_{x \in \overline{U}} \{\ell(x) - u_k(x) + \delta/2\}.$$

This maximum is attained at some $x_k \in \overline{U}$. Since $\delta_k > 0$ and $u_k(x) - \ell(x) > \delta/2$ for $x \in \partial U$, we get that $x_k \notin \partial U$. We claim that p is the slope of a supporting hyperplane to u_k at the point $(x_k, u(x_k))$. Indeed,

$$\delta_k = u(x_0) + p \cdot (x_k - x_0) - u_k(x_k) + \delta/2$$

and so

$$u_k(x) \geq u_k(x_k) + p \cdot (x - x_k) \qquad \forall x \in \overline{U}. \tag{1.2.1}$$

Since u_k is convex in Ω and U is open, (1.2.1) holds for all $x \in \Omega$, that is $p \in \partial u_k(x_k)$ for all $k \geq k_0$. This implies that $p \in \liminf_{k\to\infty} \partial u_k(U)$. ■

Lemma 1.2.3 *If u_k are convex functions in Ω such that $u_k \to u$ uniformly on compact subsets of Ω, then the associated Monge–Ampère measures Mu_k tend to Mu weakly, that is*

$$\int_\Omega f(x)\, dMu_k(x) \to \int_\Omega f(x)\, dMu(x),$$

for every f continuous with compact support in Ω.

1.3 Viscosity solutions

Definition 1.3.1 *Let $u \in C(\Omega)$ be a convex function and $f \in C(\Omega)$, $f \geq 0$. The function u is a viscosity subsolution (supersolution) of the equation* $\det D^2 u = f$ *in Ω if whenever convex $\phi \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that $(u - \phi)(x) \leq (\geq)\ (u - \phi)(x_0)$ for all x in a neighborhood of x_0, then we must have*

$$\det D^2\phi(x_0) \geq (\leq) f(x_0).$$

Remark 1.3.2 We claim that if $u \in C(\Omega)$ is convex, $\phi \in C^2(\Omega)$ and $u - \phi$ has a local maximum at $x_0 \in \Omega$, then

$$D^2\phi(x_0) \geq 0.$$

In fact, since $\phi \in C^2(\Omega)$, we have

$$\phi(x) = \phi(x_0) + D\phi(x_0) \cdot (x - x_0) + \frac{1}{2}\langle D^2\phi(x_0)(x - x_0), x - x_0\rangle + o(|x - x_0|^2).$$