NOTES ON SCHAUDER ESTIMATES

CRISTIAN E. GUTIÉRREZ

JUNE 5, 2007

Lemma 1. *If* $\Delta u \geq -f$ *in* $B_r(y)$*, then*

$$u(x) \leq \sup_{\partial B_r(y)} u + \frac{r^2 - |x-y|^2}{2n} \sup_{B_r(y)} f, \quad x \in B_r(y).$$

Proof. Let $g(x) = u(x) - \sup_{\partial B_r(y)} u - \dfrac{r^2 - |x-y|^2}{2n} \sup_{B_r(y)} f$. We have $\Delta g = \Delta u + \sup_{B_r(y)} f \geq -f + \sup_{B_r(y)} f \geq 0$, that is, g is subharmonic in $B_r(y)$. Then $\sup_{B_r(y)} g = \sup_{\partial B_r(y)} g = 0$, so $g \leq 0$ in $B_r(y)$ and the lemma follows. □

Lemma 2. *If* u *is a solution to* $\Delta u = f$ *in* $B_r(y)$ *and* v *solves* $\Delta v = 0$ *and* $v = u$ *on* $\partial B_r(y)$*, then*

$$\frac{r^2 - |x-y|^2}{2n} \inf_{B_r(y)} f \leq v(x) - u(x) \leq \frac{r^2 - |x-y|^2}{2n} \sup_{B_r(y)} f,$$

in particular,

$$|u(x) - v(x)| \leq \frac{r^2 - |x-y|^2}{2n} \sup_{B_r(y)} |f|,$$

for all $x \in B_r(y)$.

Proof. Since $\Delta(v - u) = -f$, the inequality immediately follows from Lemma 1. □

Lemma 3. *Let* $0 < \alpha < 1$. *There exist positive constants* C_0, ϵ_0 *and* $0 < \lambda < 1$ *such that for any* f *and any solution to* $\Delta u = f$ *in* $B_1(0)$ *with* $||u||_{L^\infty(B_1(0))} \leq 1$ *and* $||f||_{L^\infty(B_1(0))} \leq \epsilon_0$ *there exists a second degree harmonic polynomial* $q(x) = \frac{1}{2}\langle Ax, x\rangle + B \cdot x + C$ *such that*

$$|u(x) - q(x)| \leq \lambda^{2+\alpha}, \qquad \text{for } |x| \leq \lambda, \tag{1}$$

and

$$||A|| + |B| + |C| \leq C_0. \tag{2}$$

Proof. Let v be the harmonic function in the statement of Lemma 2. By the maximum principle $\sup_{B_1(0)} |v| \leq \sup_{B_1(0)} |u| \leq 1$. Since v is harmonic

$$|D^\beta v(x)| \leq C(n, |\beta|) \sup_{B_1(0)} |v| \leq C(n, |\beta|), \qquad \text{for } |x| \leq 1/2. \tag{3}$$

We shall prove that the second order Taylor polynomial of v about 0, $q(x) = \frac{1}{2}\langle D^2v(0)x, x\rangle + Dv(0)\cdot x + v(0)$, satisfies (1) and (2). In fact, (2) follows from (3). Also, $\Delta q(x) = \text{trace}D^2q(x) = \text{trace}D^2v(0) = \Delta v(0) = 0$, so q is harmonic. We have

$$v(x) = q(x) + \frac{1}{3!}\left[(x\cdot D)^3 v(x)\right]_{x=\xi},$$

with ξ an intermediate point between 0 and x. Then from Lemma 2 we get

$$\begin{aligned}
|u(x) - q(x)| &\leq |u(x) - v(x)| + |v(x) - q(x)| \\
&\leq \frac{1-|x|^2}{2n}\sup_{B_1(0)}|f| + \frac{1}{3!}\left|\left[(x\cdot D)^3 v(x)\right]_{x=\xi}\right| \\
&\leq \frac{1-|x|^2}{2n}\sup_{B_1(0)}|f| + C_n\,|x|^3 \sup_{|z|\leq 1/2, |\beta|=3}|D^\beta v(z)|, \qquad \text{for } |x|\leq 1/2 \\
&\leq \frac{1-|x|^2}{2n}\sup_{B_1(0)}|f| + C_n\,|x|^3 = I + II, \qquad \text{from (3).}
\end{aligned}$$

We write

$$\begin{aligned}
II = C_n\,|x|^3 &\leq C_n\,\lambda^3, \qquad \text{if } |x|\leq\lambda \\
&\leq \frac{1}{2}\,\lambda^{2+\alpha}
\end{aligned}$$

if we pick $\lambda \leq \left(\frac{1}{2C_n}\right)^{1/(1-\alpha)}$. With this value of λ, we next want

$$I \leq \frac{1}{2}\,\lambda^{2+\alpha}.$$

If $\epsilon_0 \leq n\,\lambda^{2+\alpha}$, we then have

$$I = \frac{1-|x|^2}{2n}\sup_{B_1(0)}|f| \leq \frac{1}{2n}\sup_{B_1(0)}|f| \leq \frac{1}{2n}\,\epsilon_0 \leq \frac{1}{2}\,\lambda^{2+\alpha}$$

and we are done. □

Theorem 4. *Suppose* $u \in C^2(B_1(0)) \cap C(\bar{B}_1(0))$, $\Delta u = f$, *and* f *is Hölder continuous at* 0, *i.e.,*

$$[f]_{\alpha,0} = \sup_{|x|\leq 1}\frac{|f(x) - f(0)|}{|x|^\alpha} < \infty.$$

Then there exists a second degree polynomial $p(x,0) = \frac{1}{2}\langle Ax, x\rangle + B\cdot x + C$ *such that*

$$|u(x) - p(x,0)| \leq C_1\,|x|^{2+\alpha}, \qquad \text{for } |x|\leq 1/2, \tag{4}$$

with

$$C_1 \leq C_0\left([f]_{\alpha,0} + \|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}\right),$$

and

$$\|A\| + |B| + |C| \leq C_0 \left([f]_{\alpha,0} + \|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right).$$

Proof. We may assume that

(i) $f(0) = 0$,

(ii) $[f]_{\alpha,0} + \|f\|_{L^\infty(B_1)} \leq \epsilon_0$,

(iii) $\|u\|_{L^\infty(B_1(0)} \leq 1$.

Indeed, if we let

$$v(x) = u(x) - \frac{|x|^2}{2n} f(0), \qquad h(x) = \epsilon_0 \frac{f(x) - f(0)}{[f]_{\alpha,0} + 2\|f\|_{L^\infty(B_1)} + \|v\|_{L^\infty(B_1)}}$$

and

$$\bar{u}(x) = \epsilon_0 \frac{v(x)}{[f]_{\alpha,0} + 2\|f\|_{L^\infty(B_1)} + \|v\|_{L^\infty(B_1)}},$$

then $\Delta \bar{u} = h$ in B_1, h satisfies (i) and (ii), and $\bar{u}$ satisfies (iii).

Claim: there exists a sequence of harmonic polynomials $p_k(x) = \frac{1}{2}\langle A_k x, x\rangle + B_k \cdot x + C_k$ such that

$$|u(x) - p_k(x)| \leq \lambda^{(2+\alpha)k}, \qquad \text{for } |x| \leq \lambda^k \tag{5}$$

and

$$\|A_k - A_{k+1}\| \leq C\,\lambda^{\alpha k}, \quad |B_k - B_{k+1}| \leq C\,\lambda^{(\alpha+1)k}, \quad |C_k - C_{k+1}| \leq C\,\lambda^{(\alpha+2)k}, \tag{6}$$

for $k = 1, \cdots$ where C is a universal constant. In view of (ii) and (iii) above we can apply Lemma 3, and we let $p_1(x)$ be the polynomial in that lemma. Suppose $p_k(x)$ is constructed. We will construct p_{k+1}. Let

$$w(x) = \frac{(u - p_k)(\lambda^k x)}{\lambda^{(\alpha+2)k}}.$$

We have

$$\Delta w(x) = \frac{1}{\lambda^{(\alpha+2)k}} \left[\lambda^{2k} (\Delta u)(\lambda^k x) - \lambda^{2k} (\Delta p_k)(\lambda^k x) \right] = \frac{1}{\lambda^{\alpha k}} f(\lambda^k x) = g_k(x).$$

From (5), $\|w\|_{L^\infty(B_1)} \leq 1$ and from (i) and (ii) above, $\|g_k\|_{L^\infty(B_1)} \leq \epsilon_0$. Hence by application of Lemma 3 to w, we get a harmonic polynomial $q_k(x)$ -depending on g_k- such that

$$|w(x) - q_k(x)| \leq \lambda^{2+\alpha}, \qquad \text{for } |x| \leq \lambda. \tag{7}$$

From the definition of w and (7) we then get

$$|u(\lambda^k x) - p_k(\lambda^k x) - \lambda^{(2+\alpha)k} q_k(x)| \leq \lambda^{(2+\alpha)(k+1)}, \quad \text{for } |x| \leq \lambda.$$

Therefore, if we take

$$p_{k+1}(x) = p_k(x) + \lambda^{(2+\alpha)k} q_k(x/\lambda^k), \tag{8}$$

then p_{k+1} satisfies (5) with k replaced by $k+1$. Writing $q_k(x) = \frac{1}{2}\langle A_k^* x, x\rangle + B_k^* \cdot x + C_k^*$, from (8) we get

$$A_{k+1} = A_k + \lambda^{\alpha k} A_k^*, \quad B_{k+1} = B_k + \lambda^{(\alpha+1)k} B_k^*, \quad C_{k+1} = C_k + \lambda^{(\alpha+2)k} C_k^*,$$

for $k = 1, 2, \cdots$ and then (6) follows from (2). This completes the proof of the claim.

Next, we notice that from (6), (2), and since $0 < \lambda < 1$, it follows that A_k, B_k, C_k are Cauchy sequences and therefore we let

$$p(x, 0) = \frac{1}{2}\langle A_\infty x, x\rangle + B_\infty \cdot x + C_\infty,$$

where $A_\infty, B_\infty, C_\infty$ are the corresponding limits. We show that $p(x, 0)$ satisfies (4). Given $|x| \leq 1/2$, let k be a positive integer such that $\lambda^{k+1} < |x| \leq \lambda^k$. Hence

$$|p_k(x) - p(x, 0)| \leq C\,(\lambda^{\alpha k}|x|^2 + \lambda^{(\alpha+1)k}|x| + \lambda^{(\alpha+2)k} \approx C\,|x|^{2+\alpha}.$$

and from (5) we obtain

$$|u(x) - p(x, 0)| \leq |u(x) - p_k(x)| + |p_k(x) - p(x, 0)| \leq C\,|x|^{2+\alpha} + C\,|x|^{2+\alpha}$$

and we are done. □

Suppose $f \in C^\alpha(B_1(0))$ and u is a solution to $\Delta u = f$ in $B_1(0)$. Let $y \in B_1(0)$ and $r < \operatorname{dist}(y, \partial B_1(0))$. Define $g(x) = r^2 f(y + rx)$ and $v(x) = u(y + rx)$ for $x \in B_1(0)$. We have that v is a solution to $\Delta v = g$ in $B_1(0)$ and

$$[g]_{\alpha,0} = \sup_{|x|\leq 1} \frac{|g(x) - g(0)|}{|x|^\alpha} = r^2 \sup_{|x|\leq 1} \frac{|f(y + rx) - f(y)|}{|x|^\alpha} = r^{2+\alpha} \sup_{|z|\leq r} \frac{|f(y + z) - f(y)|}{|z|^\alpha}.$$

From Theorem 4 applied to v, there exists a quadratic polynomial $p(x, 0)$ such that

$$|v(x) - p(x, 0)| \leq C_1\,|x|^{2+\alpha}, \qquad \text{for } |x| \leq 1/2, \tag{9}$$

with

$$C_1 \leq C_0 \left([g]_{\alpha,0} + \|g\|_{L^\infty(B_1(0))} + \|v\|_{L^\infty(B_1(0))}\right). \tag{10}$$

From (9) and the definition of v we get that

$$|u(z) - p((z - y)/r, 0)| \leq \frac{C_1}{r^{2+\alpha}}\,|z - y|^{2+\alpha}, \qquad \text{for } |z - y| \leq r/2.$$

We have $\|g\|_{L^\infty(B_1(0))} = r^2 \|f\|_{L^\infty(B_r(y))}$, and $\|v\|_{L^\infty(B_1(0))} = \|u\|_{L^\infty(B_r(y))}$. If we let $q(x,y) = p((x-y)/r, 0)$, and $r = \operatorname{dist}(y, \partial B_1(0))$, then

$$|u(x) - q(x,y)| \le C_1^* |x-y|^{2+\alpha}, \qquad \text{for } |x-y| \le \operatorname{dist}(y, \partial B_1(0))/2,$$

with

$$\begin{aligned} C_1^* &= \frac{C_1}{\operatorname{dist}(y, \partial B_1(0))^{2+\alpha}} \\ &\le C_0 \Bigg(\sup_{|z| \le \operatorname{dist}(y,\partial B_1(0))} \frac{|f(y+z) - f(y)|}{|z|^\alpha} \\ &\qquad + \operatorname{dist}(y, \partial B_1(0))^{-\alpha} \|f\|_{L^\infty(B_r(y))} + \operatorname{dist}(y, \partial B_1(0))^{-2-\alpha} \|u\|_{L^\infty(B_r(y))} \Bigg). \end{aligned}$$

In particular, we obtain that if $\Delta u = f$ in $B_1(0)$, then for each $y \in B_{1/2}(0)$ there exists a quadratic polynomial $p(x,y)$ such that

$$|u(x) - p(x,y)| \le C^* |x-y|^{2+\alpha}, \qquad \text{for } |x-y| \le 1/4,$$

with

$$C^* \le C_0 \left([f]_{\alpha, B_1(0)} + \|f\|_{L^\infty(B_1(0))} + \|u\|_{L^\infty(B_1(0))} \right).$$

Lemma 5. *Suppose $u \in C^2(\Omega)$ is such that there exist constants $C_1 > 0$ and $0 < \alpha < 1$ so that for each $y \in \Omega$ there exists a quadratic polynomial $p(x,y)$ such that*

$$|u(x) - p(x,y)| \le C_1 |x-y|^{2+\alpha}, \qquad \text{for all } x \in B(y, \operatorname{dist}(y, \partial\Omega)/2). \tag{11}$$

Then

$$p(x,y) = \frac{1}{2}(x-y)^t D^2 u(y)(x-y) + Du(y)\cdot(x-y) + u(y), \tag{12}$$

and

$$|D_{ij}u(x_1) - D_{ij}u(x_2)| \le C_1 |x_1 - x_2|^\alpha \tag{13}$$

for all $x_1, x_2 \in \Omega$ with $\operatorname{dist}(x_i, \partial\Omega) > \operatorname{diam}(\Omega)/2$.

Proof. We use the following result: if $f \in C^2(I)$ where I is an open interval, then

$$\lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a), \tag{14}$$

for $a \in I$. (Notice that the converse to this result is not true, take $f(x) = x|x|$ at $a = 0$).

From (11) it follows immediately that $p(y,y) = u(y)$. On the other hand,

$$D_j u(y) = \lim_{h\to 0} \frac{u(he_j + y) - u(y)}{h},$$

and

$$\left|\frac{u(h\,e_j+y)-u(y)}{h}-\frac{p(h\,e_j+y,y)-p(y,y)}{h}\right|\leq\frac{C_1\,|h\,e_j|^{2+\alpha}}{|h|}\to 0,$$

as $h\to 0$. So $D_xp(y,y)=Du(y)$. If η is a nonzero vector in R^n and $g_\eta(t)=u(t\,\eta+y)$, then $g_\eta''(t)=\sum_{i,j=1}^n\eta_i\eta_j u_{ij}(t\,\eta+y)$. In particular, $g_{e_k}''(0)=u_{kk}(y)$, and $g_{e_k+e_\ell}''(0)=u_{kk}(y)+2u_{k\ell}(y)+u_{\ell\ell}(y)$. Hence

$$u_{k\ell}(y)=\frac{1}{2}\left\{g_{e_k+e_\ell}''(0)-g_{e_k}''(0)-g_{e_\ell}''(0)\right\},$$

and from (14) we get

$$u_{k\ell}(y)=\frac{1}{2}\lim_{h\to 0}\frac{\Delta_{k\ell}u(h,y)+\Delta_{k\ell}u(-h,y)-2\Delta_{k\ell}u(0,y)}{h^2}$$

where

$$\Delta_{k\ell}u(h,y)=u(h(e_k+e_\ell)+y)-u(he_k+y)-u(he_\ell+y).$$

Also

$$p_{k\ell}(y)=\frac{1}{2}\lim_{h\to 0}\frac{\Delta_{k\ell}p(h,y)+\Delta_{k\ell}p(-h,y)-2\Delta_{k\ell}p(0,y)}{h^2}$$

Since

$$|\Delta_{k\ell}u(h,y)-(p(h(e_k+e_\ell)+y,y)-p(he_k+y,y)-p(he_\ell+y,y))|\leq 4\,C_1\,|h|^{2+\alpha},$$

$$|\Delta_{k\ell}u(-h,y)-(p(-h(e_k+e_\ell)+y,y)-p(-he_k+y,y)-p(-he_\ell+y,y))|\leq 4\,C_1\,|h|^{2+\alpha},$$

and $\Delta_{k\ell}u(0,y)=-u(y)=-p(y,y)$ we obtain that $D^2u(y)=D_x^2p(y,y)$ and then (12) is proved.

To prove (13) we use the following lemma of Calderón-Zygmund, [CZ61, Lemma 2.6].

Lemma 6. *Given an integer $m\geq 0$, there exists a function $\varphi\in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball such that $\varphi_\epsilon * P=P$ for each $\epsilon>0$ and every polynomial P of degree $\leq m$. As usual, $\varphi_\epsilon(x)=\epsilon^{-n}\varphi(x/\epsilon)$.*

Proof. Let C be the class of C^∞ functions in $\mathbb{R}^n$ with support in the unit ball, and define

$$T(\phi)=\int_{\mathbb{R}^n}\phi(x)\,x^\alpha\,dx.$$

The linear transformation T maps C into the vector space V of all points $\{\xi_\alpha\}$ with $0\leq|\alpha|\leq m$. We claim that this map is onto. Otherwise, $T(C)$ is a subspace strictly contained in V, and so $T(C)$ has an orthogonal complement $V' \neq \{0\}$ in V. Therefore there exists $\{\eta_\alpha\} \in V'$ not all zero with

$$\sum \eta_\alpha \xi_\alpha = \int_{\mathbb{R}^n} \phi(x) \sum \eta_\alpha x^\alpha \, dx = 0$$

for all $\phi \in C$. In particular,if $\psi \in C$ is any function with $\psi > 0$ for $|x| > 1$, then taking $\phi(x) = \psi(x) \sum \bar{\eta}_\alpha x^\alpha$, we obtain

$$\int_{\mathbb{R}^n} \psi(x) \left| \sum \eta_\alpha x^\alpha \right|^2 \, dx = 0$$

and consequently $\left|\sum \eta_\alpha x^\alpha\right|$=0 for $|x| < 1$ and therefore η_α are all zero, a contradiction. Therefore $T(C) = V$ and so there exists $\phi \in C$ such that $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$ and $\int_{\mathbb{R}^n} \phi(x)\, x^\alpha \, dx = 0$ for $0 < |\alpha| \leq m$. If $Q(x)$ is a polynomial of degree $\leq m$, then $\int_{\mathbb{R}^n} \phi(x)\, Q(x)\, dx = Q(0)$. Therefore, $\epsilon^{-n} \int_{\mathbb{R}^n} \phi((x-y)/\epsilon)\, Q(y)\, dy = \int_{\mathbb{R}^n} \phi(z)\, Q(x + \epsilon z)\, dz = Q(x)$. □

Let $x_1, x_2 \in \Omega$ be such that $\operatorname{dist}(x_i, \partial\Omega) > \operatorname{diam}(\Omega)/2$ and write

$$\begin{aligned} u(x) &= u(x) - p(x, x_1) + p(x, x_1) \\ u(x) &= u(x) - p(x, x_2) + p(x, x_2), \end{aligned}$$

and convolving these expressions with φ_ϵ and using Lemma 6 with $m = 2$ we get

$$\begin{aligned} u_\epsilon(x) &= [u - p(\cdot, x_1)] * \varphi_\epsilon(x) + p(x, x_1) \\ u_\epsilon(x) &= [u - p(\cdot, x_2)] * \varphi_\epsilon(x) + p(x, x_2), \end{aligned}$$

for $\operatorname{dist}(x, \partial\Omega) > \epsilon$, and taking derivatives

$$\begin{aligned} D_{ij} u_\epsilon(x) &= [u - p(\cdot, x_1)] * D_{ij}\varphi_\epsilon(x) + D_{ij} u(x_1) \\ D_{ij} u_\epsilon(x) &= [u - p(\cdot, x_2)] * D_{ij}\varphi_\epsilon(x) + D_{ij} u(x_2). \end{aligned}$$

Hence

$$\begin{aligned} D_{ij} u(x_1) - D_{ij} u(x_2) &= [u - p(\cdot, x_2)] * D_{ij}\varphi_\epsilon(x) - [u - p(\cdot, x_1)] * D_{ij}\varphi_\epsilon(x) \\ &= \epsilon^{-n-2} \int_{|y-x|<\epsilon} [u(y) - p(y, x_2)] D_{ij}\varphi((x-y)/\epsilon)\, dy \\ &\quad - \epsilon^{-n-2} \int_{|y-x|<\epsilon} [u(y) - p(y, x_1)] D_{ij}\varphi((x-y)/\epsilon)\, dy \\ &= I - II. \end{aligned}$$

If we let $x = (x_1 + x_2)/2$, and $\epsilon = |x_1 - x_2|/2$, we get that $B_\epsilon(x) \subset B_{2\epsilon}(x_i)$ for $i = 1, 2$, and so from (11) we get

$$\begin{aligned}|I| &\leq \epsilon^{-n-2} C_1 \int_{|y-x_2|<2\epsilon} |y - x_2|^{2+\alpha} |D_{ij}\varphi((x-y)/\epsilon)|\, dy \\ &\leq \epsilon^{-n-2} C_1\, \|D_{ij}\varphi\|_\infty \int_{|y-x_2|<2\epsilon} |y - x_2|^{2+\alpha}\, dy = C_n\, C_1\, \epsilon^\alpha = C\, |x_1 - x_2|^\alpha,\end{aligned}$$

and

$$\begin{aligned}|II| &\leq \epsilon^{-n-2} C_1 \int_{|y-x_1|<2\epsilon} |y - x_1|^{2+\alpha} |D_{ij}\varphi((x-y)/\epsilon)|\, dy \\ &\leq \epsilon^{-n-2} C_1\, \|D_{ij}\varphi\|_\infty \int_{|y-x_1|<2\epsilon} |y - x_1|^{2+\alpha}\, dy = C_n\, C_1\, \epsilon^\alpha = C\, |x_1 - x_2|^\alpha,\end{aligned}$$

and (13) follows. □

References

[CZ61] A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math. **20** (1961), 171–225.

Department of Mathematics, Temple University, Philadelphia, PA 19122
E-mail address: `gutierrez@math.temple.edu`