Fall 2006

Test 2

Math 127

Department of Mathematics
Temple University

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Name: ANSWER KEY

Instructor/Section: GUTIÉRREZ/2^2

This exam consists of 9 questions. Show all your work. **No work - no credit.** Good Luck!

Question	Points	Out of
1		16
2		13
3		12
4		10
5		12
6		15
7		10
8		12
9		4
Total		**104**

16pt **1.** Let $f(x,y,z) = yze^{-x^2y^2}$.

(**a**) Find the first partial derivatives of f.

$f_x = -2xy^3ze^{-x^2y^2}$; $f_y = ze^{-x^2y^2} - 2y^2x^2ze^{-x^2y^2}$; $f_z = ye^{-x^2y^2}$.

(**b**) Find the gradient of f at the point $P(0,2,3)$.

$\nabla f(0,2,3) = 0\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

(**c**) Find the directional derivative of f at $P(0,2,3)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$.

$|\mathbf{v}| = \sqrt{4+36+9} = \sqrt{49} = 7$; $\mathbf{u} = \mathbf{v}/|\mathbf{v}| = 2/7\mathbf{i} - 6/7\mathbf{j} + 3/7\mathbf{k}$; $\partial_{\mathbf{u}} f(0,2,3) = \nabla f(0,2,3) \cdot u = -18/7 + 6/7 = -12/7$.

(**d**) In which direction is f increasing most rapidly at $P(0,2,3)$? What is the rate of this increase?

In the direction of the gradient at that point: $0\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Rate equals $|\nabla f(0,2,3)| = \sqrt{13}$.

13pt **2.** Let $f(x,y,z) = xy + xz^2$, where $x = u+v$, $y = u-v$, and $z = uv$. Find the partial derivatives $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ when $u = 1$, $v = -2$.

$\dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial u} = (y+z^2)1 + x1 + 2xzv$. If $u = 1$ and $v = -2$, then $x = -1$, $y = 3$ and $z = -2$. So at that point $\dfrac{\partial f}{\partial u} = -2$. Similarly, $\dfrac{\partial f}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial v} = (y+z^2)1 + x(-1) + 2xzu$; and $\dfrac{\partial f}{\partial v} = 12$.

12pt **3.** Find the local maximum and minimum values and saddle points of the function

$f(x,y) = x^2 \ln y - y$.

$f_x = 2x\ln y$; $f_y = x^2/y - 1$. Solve $2x\ln y = 0$ and $x^2/y - 1 = 0$, which yields the solutions $(1,1)$ and $(-1,1)$. Also $f_{xx} = 2\ln y$; $f_{xy} = 2x/y$; $f_{yy} = -x^2/y^2$. At $(1,1)$, $f_{xx} = 0$, $f_{xy} = 2$, $f_{yy} = -1$, so $D = 0(-1) - 2.2 = -4$ and therefore $(1,1)$ is a saddle point. At $(-1,1)$, $f_{xx} = 0$, $f_{xy} = -2$, $f_{yy} = -1$, so $D = 0(-1) - (-2).(-2) = -4$ and therefore $(-1,1)$ is also a saddle point.

10pt **4.** Sketch the region of integration and change the order of integration.

$$\int_0^2 \int_{x^2}^{2x} f(x,y)\, dy\, dx$$

The region is the one bounded by the line $y = 2x$ and the parabola $y = x^2$.

$$\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\, dx\, dy.$$

12pt **5.** Evaluate using polar coordinates.

$$\iint_D \sin(x^2+y^2)\,dA, \quad \text{where} \quad D = \{(x,y) \mid -2 \le x \le 0,\ x^2+y^2 \le 4\}$$

In polar coordinates the region is $D = \{(r,\theta) : 0 \le r \le 2, \pi/2 \le \theta \le 3\pi/2\}$.

So $\iint_D \sin(x^2+y^2)\,dA = \int_0^2 \int_{\pi/2}^{3\pi/2} r\,\sin(r^2)\,d\theta\,dr = \pi \int_0^2 r\,\sin(r^2)\,dr = \pi\left(-\frac{\cos(r^2)}{2}\right)\Big|_{r=0}^{r=2} =$ $\pi/2(1-\cos 4)$.

15pt **6.** Evaluate the integral $\iiint_E z\,dV$, where E is the solid bounded above by the surface $z = 2\sqrt{xy}$ and lying above the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$ in the xy-plane.

The region is $E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 2\sqrt{xy}\}$

So $\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{2\sqrt{xy}} z\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \int_0^1 2x \int_0^{1-x} y\,dy\,dx = \int_0^1 x(1-$ $x)^2\,dx = 1/12$.

10pt **7.** Let E be the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 4$. Use **spherical** coordinates to express the volume of E as an iterated integral. **Do not** evaluate the integral.

The surfaces intersect when $(\sqrt{x^2+y^2})^2 + x^2 + y^2 = 4$, that is, $x^2+y^2 = 2$, i.e., the circle of radius $\sqrt{2}$. In spherical coordinates the region is $E = \{(\rho,\theta,\phi) : 0 \le \rho \le \sqrt{2}, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/4\}$.

So Volume(E)=$\int_0^{\sqrt{2}} \int_0^{2\pi} \int_0^{\pi/4} \rho^2\,\sin\phi\,d\phi\,d\theta\,d\rho$.

12pt **8.** Evaluate the iterated integral by converting it to **cylindrical** coordinates.

$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} y\,dz\,dy\,dx$$

In cylindrical coordinates the region is $E = \{(r,\theta,z) : 0 \le r \le 3, 0 \le \theta \le \pi, 0 \le z \le r\}$. Then $\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} y\,dz\,dy\,dx = \int_0^3 \int_0^{\pi} \int_0^{r} r^2 \sin\theta\,dz\,d\theta\,dr = \int_0^3 \int_0^{\pi} r^3 \sin\theta\,d\theta\,dr = \int_0^3 r^3\,dr \int_0^{\pi} \sin\theta\,d\theta =$ $(3^4/4)\,2 = 81/2$.

4pt **9.** True of false? (You don't need to explain your answers.)

(**a**) $\int_1^3 \int_2^5 f(x,y)\,dy\,dx = \int_2^5 \int_1^3 f(x,y)\,dx\,dy$ TRUE (integration over a rectangle)

(**b**) $\int_1^3 \int_2^x f(x,y)\,dy\,dx = \int_2^x \int_1^3 f(x,y)\,dx\,dy$ FALSE (the second integral gives a function of x)

(**c**) $\int_1^3 \int_2^x f(x,y)\,dy\,dx = \int_1^3 \int_2^y f(y,x)\,dx\,dy$ TRUE (the variables x and y are switched)

(**d**) $\int_1^3 \int_2^5 f(x)\cdot g(y)\,dy\,dx = \int_1^3 f(x)\,dx \cdot \int_2^5 g(y)\,dy$ TRUE (the integrand has separate variables)