Math 2043 — Summer I 2009 — First Exam

Department of Mathematics

Temple University

June 3, 2009

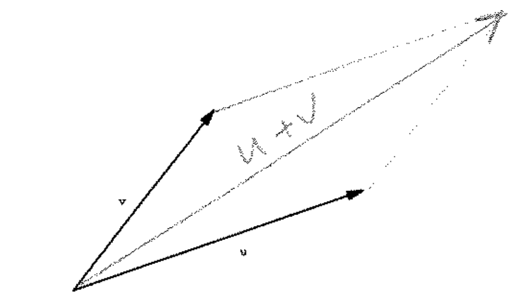
Name: ______________________________

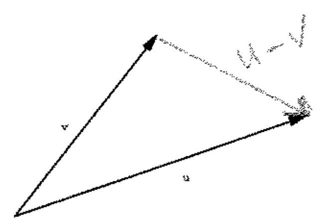
Instructor/Section: ______________________________

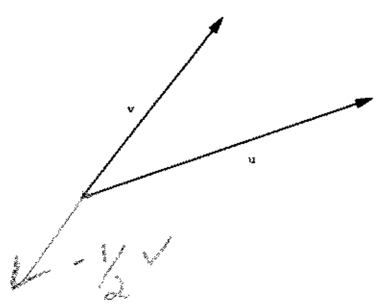
This exam consists of 10 questions. Show all your work. **No work, no credit.** Good Luck!

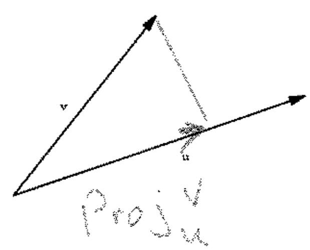
Question	Points	Out of
1		8
2		16
3		12
4		8
5		5
6		10
7		14
8		14
9		5
10		13
Total		**105**

1. (8 points) On the four diagrams below you see two vectors, $\mathbf{u}$ and $\mathbf{v}$. On the same diagrams sketch the vectors $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $-\frac{1}{2}\mathbf{v}$, and the vector projection $\text{proj}_{\mathbf{u}}\mathbf{v}$. Make sure to sketch one vector per diagram and in the order in which they are listed above.









2. (16 points) Let $\mathbf{u} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = 4\mathbf{i} - \mathbf{k}$ Find

(a) $2\mathbf{u} + 3\mathbf{v}$

$$= 6i - 2j + 8k + 12i - 3k = 18i - 2j + 5k$$

(b) The unit vector in the direction opposite to $\mathbf{u}$.

$$\|u\| = \sqrt{26} \qquad v = -\frac{u}{\|u\|} = -\frac{3}{\sqrt{26}}i + \frac{1}{\sqrt{26}}j - \frac{4}{\sqrt{26}}k$$

(c) $\mathbf{u} \cdot \mathbf{v}$.

$$= 3 \cdot 4 + 4 \cdot (-1) = 8$$

(d) The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos\theta = \frac{8}{\sqrt{17}\sqrt{26}}$$

(e) The vector projection $\text{proj}_{\mathbf{u}}\mathbf{v}$.

$$\text{proj}_u v = \frac{u \cdot v}{\|u\|^2} u = \frac{8}{26}(3i - j + 4k) = \frac{24}{26}i - \frac{8}{26}j + \frac{32}{26}$$

(f) All values of a that make the vector $a\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ orthogonal to $\mathbf{u}$

$$u \cdot (ai + 2j + 5k) = 3a - 2 + 20 = 0 \qquad 3a = -18$$

$$a = -6$$

(g) All values of b and c which make the vector $2\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ parallel to $\mathbf{v}$.

$$2i + bj + ck = \frac{1}{2}(4i - k) \qquad b = 0 \qquad c = -\tfrac{1}{2}$$

3. (12 points) Let $P(3,-1,2)$, $Q(6,2,4)$, and $R(4,2,1)$. Find

(a) The components of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$\overrightarrow{PQ} = \langle 3, 3, 2 \rangle$

$\overrightarrow{PR} = \langle 1, 3, -1 \rangle$

(b) Parametric equations of the line through P and Q.

$x = 3 + 3t, \quad y = -1 + 3t, \quad z = 2 + 2t$

(c) $\overrightarrow{PQ} \times \overrightarrow{PR}$

$$= \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 3 & 3 & 2 \\ 1 & 3 & -1 \end{vmatrix} = \langle -9, 5, 6 \rangle$$

(d) An equation of the plane through the points P, Q, and R. Please give your answer in the form $ax + by + cz = d$.

$-9x + 5y + 6z = -27 - 5 + 12 = -20$

or

$9x - 5y - 6z = 20$

4. (8 points) Let $\mathbf{u} = \langle 6, 3, -1 \rangle$, $\mathbf{v} = \langle 0, 1, 2 \rangle$ and $\mathbf{w} = \langle 4, -2, -5 \rangle$.

(a) Find the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

$$u \times v = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 6 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \langle 7, -12, 6 \rangle$$

$$A = \sqrt{49 + 144 + 36} = \sqrt{229}$$

(b) Find the volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$$V = |w \cdot (u \times v)| = |28 + 24 - 30| = 22$$

5. (5 points) Find the point of intersection of the line $\mathbf{r}(t) = \langle 2, 3, 0 \rangle + t\langle 2, -1, 1 \rangle$ and the plane $2x + 3y + z = 9$.

$$2(2+2t) + 3(3-t) + t = 9$$

$$4 + 4t + 9 - 3t + t = 9$$

$$2t = -4 \qquad t = -2$$

$$P = (-2, 5, -2)$$

6. (10 points) Find $\mathbf{r}(t)$ that satisfies $\frac{d\mathbf{r}}{dt} = \langle e^{-t}, \frac{t}{t^2+1}, \sin 2t\rangle$, $\mathbf{r}(0) = \langle 2, 1, 0\rangle$.

$$r(t) = \int \langle e^{-t}, \frac{t}{t^2+1}, \sin 2t \rangle \, dt$$

$$= \langle -e^{-t}, \frac{1}{2}\ln(t^2+1), -\frac{\cos 2t}{2} \rangle + C$$

$$\langle 2, 1, 0 \rangle = \langle -1, 0, -\tfrac{1}{2} \rangle + C$$

$$C = \langle 3, 1, \tfrac{1}{2} \rangle \qquad r(t) = \langle 3 - e^{-t}, 1 + \tfrac{1}{2}\ln(t^2+1), \frac{1-\cos 2t}{2} \rangle$$

7. (14 points) Let $\mathbf{r}(t) = \langle 1, 2t^2, t^3\rangle$. Find

(a) A parametrization of the tangent line to the curve $\mathbf{r}(t)$ at $t = 1$.

$$r'(t) = \langle 0, 4t, 3t^2 \rangle \qquad r'(1) = \langle 0, 4, 3 \rangle$$

$$r(1) = \langle 1, 2, 1 \rangle$$

$$L(t) = \langle 1, 2, 1 \rangle + t\langle 0, 4, 3 \rangle$$

(b) The length of the curve $\mathbf{r}(t)$, $0 \le t \le 1$.

$$L = \int_0^1 \|r'(t)\| \, dt = \int_0^1 \sqrt{16t^2 + 9t^4} \, dt$$

$$= \int_0^1 t\sqrt{16 + 9t^2} \, dt = \frac{2(16+9t^2)^{3/2}}{54} \Big|_0^1$$

$$= \frac{125 - 64}{27} = \frac{61}{27}$$

8. (14 points) Let $f(x, y) = e^{-y/x^2}$.

(a) Find the domain of $f(x, y)$.

$x \neq 0$

(b) Find the range of $f(x, y)$.

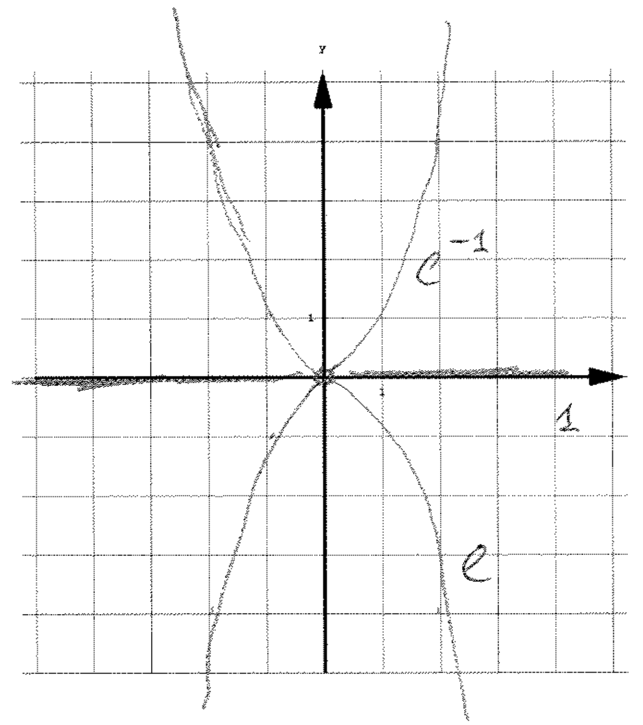
$(0, \infty)$

(c) Determine the equations and sketch the level curves $f(x, y) = k$ for $k = e^{-1}$, 1, e on the grid provided. Please **label** each curve by writing $k = e^{-1}$, $k = 1$, or $k = e$ next to it.

$e^{-y/x^2} = e^{-1}$ $\quad y = x^2$

$e^{-y/x^2} = 1$ $\quad y = 0$

$e^{-y/x^2} = e$ $\quad y = -x^2$



(d) Evaluate $\lim_{(x,y)\to(0,1)} e^{-y/x^2}$ or determine that the limit does not exist. Be sure to justify your answer.

As $(x, y) \to (0, 1)$, $-y/x^2 \to -\infty$

and $e^{-y/x^2} \to 0$

$\lim_{(x,y)\to(0,1)} e^{-y/x^2} = 0$

9. (5 points) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

As $(x,y) \to (0,0)$ along $y = mx$

$$\frac{x^2-y^2}{x^2+y^2} = \frac{x^2 - m^2x^2}{x^2+m^2x^2} = \frac{1-m^2}{1+m^2} \to \frac{1-m^2}{1+m^2}$$

This answer depends on m, so the limit does not exist.

10. (13 points) (a) Compute the first order partial derivatives of $z = y^2 e^{-x/y}$

$$\frac{\partial z}{\partial x} = y^2 e^{-x/y} \cdot (-1/y) = -y e^{-x/y}$$

$$\frac{\partial z}{\partial y} = 2y e^{-x/y} + y^2 e^{-x/y} \cdot \left(\frac{x}{y^2}\right)$$

$$= 2y e^{-x/y} + x e^{-x/y}$$

(b) Compute f_{uuv} for $f(u, v) = \ln(v^2 - u)$.

$$f_u = \frac{-1}{v^2-u}$$

$$f_{uu} = \frac{-1}{(v^2-u)^2}$$

$$f_{uuv} = \frac{(-1)(-2) \cdot 2v}{(v^2-u)^3} = \frac{4v}{(v^2-u)^3}$$