Fourier Coefficients in Vector Valued Modular Forms of positive dimension

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Knopp and Mason [2] obtained growth conditions of the Fourier coefficients of vector-valued modular forms of negative dimension. In [3] they developed a general theory of vector-valued modular forms.

Let $F(\tau) = (F_1(\tau), \ldots, F_p(\tau))^t$ be a p-tuple of functions holomorphic in the complex upper half-plane $\mathcal{H}$ and $\rho : \Gamma \longrightarrow GL(p, \mathbb{C})$ a p-dimensional complex representation.(F, ρ), or simply F, is a vector-valued form of real dimension r on the modular group $\Gamma = SL(2, \mathbb{Z})$ if

1. For all $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$(F_1(\tau), \ldots, F_p(\tau)) \mid_r V(\tau) = \rho(V)(F_1(\tau), \ldots, F_p(\tau)) \tag{1}$$

2. Each component function $F_j(\tau)$ has a convergent x-expansion meromorphic at infinity:

$$F_j(\tau) = x^{m_j} \sum_{\nu \geq \mu_j} a_\nu(j) x^\nu \tag{2}$$

with $0 \leq m_j < 1$ a positive rational number, μ_j an integer and $x = e^{2\pi i \tau}$. The Slash operator $|_r V$ in (1) is defined by:

$$F \mid_r V(\tau) = F \mid_r^\varepsilon V(\tau) = \varepsilon(V)^{-1} (c\tau + d)^r F(V\tau). \tag{3}$$

In the rest of the paper we will only be interested in the elements of Γ of the form

$$V = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix}, \quad k > 0, \quad 0 \leq h, h' < k \tag{4}$$

Note that since $V \in \Gamma$, we have that $hh' \equiv -1 \pmod{k}$, and therefore $-\frac{hh'+1}{k}$ will be an integer, and it always exists an h' such that $0 \leq h' < k$, and it is unique and therefore V is uniquely determined by h and k, i.e. $V = V_{k,h}$. To simplify the computations later it will be interesting to change the multiplicative system ε for a new quantity v such that

$$F \mid_r^v V_{k,h}(\tau) = v(V_{k,h})^{-1} \left(-i(k\tau - h)\right)^r F(V_{k,h}\tau) \tag{5}$$

but first we have to make sure that we can do that. Since $k > 0$, we have that

$$0 < \arg(k\tau - h) < \pi, \quad -\frac{\pi}{2} < \arg -i(k\tau - h) < \frac{\pi}{2} \tag{6}$$

and therefore

$$arg(k\tau - h) - \frac{\pi}{2} = arg - i(k\tau - h) \tag{7}$$

and we can define $v(V) = \varepsilon(V)(-i)^{-r}$

Also given a vector-valued modular form (F, ρ), we can define

$$f_j(x) = x^{-m_j} F_j(\tau) = \sum_{\nu \geq \mu_j}^{\infty} a_\nu(j) x^\nu, \quad 1 \leq j \leq p \tag{8}$$

which are analytic in the unit circle and have a pole at $x = 0$ of order $-\mu_j$, provided that μ_j is negative. We will use this function to get the coefficients of the vector-valued modular form $a_\nu(j)$ by using Cauchy's formula. But to do so we will need later in the paper a bound for $\rho(V)$.

To bound $\rho(V)$ we introduce the Eichler length of V [1] with respect to the generators S, T of Γ. Namely we write V as a product $V = \pm V_1 \dots V_L$, where each V_L is equal to either S or T^{n_j}, for some integer n_j, no two consecutive V_j are both equal to S or a power of T, and where L is minimal. Eichler proved that

$$L(V) \leq n_1 \log \mu(V) + n_2, \tag{9}$$

where $\mu(V) = {h'}^2 + \left(\frac{hh'+1}{k}\right)^2 + k^2 + h^2$, and n_1, n_2 are constants independent of V. Also Knopp and Mason [2] showed that

$$\mid \rho_{jm}(V) \mid \leq p^{L(V)-1} K_1^{L(V)}, \quad 1 \leq m, j \leq p \tag{10}$$

where K_1 is a constant that satisfies $|\ \rho_{jm}(S)\ | \leq K_1$ for all $1 \leq m, j \leq p$. Therefore if we use the usual norm

$$|\ \rho(V)\ | = \sqrt{\sum_{1\leq m,j\leq p} |\ \rho_{jm}(V)\ |^2} \tag{11}$$

we see that

$$|\ \rho(V)\ | \leq K_2 \mu(V)^{\alpha}, \quad \alpha = n_1 \log pK_1 \tag{12}$$

where K_2 is independent of V. Note that since the length of V^{-1} is the same as the length of V, we can use the same bound for $|\ \rho(V)^{-1}\ | = |\ \rho(V^{-1})\ |$

Theorem 1. *Let (F, ρ) be a vector-valued modular form of dimension $r > 2\alpha$, where*

$$F_j(\tau) = x^{m_j} \sum_{\nu \geq \mu_j}^{\infty} a_\nu(j) x^\nu, \tag{13}$$

where α is given by the Eichler estimate(12), then the coefficients $a_\nu(j)$ are independent of $a_0(j)$, $a_1(j)$, ...

To prove the theorem we will use the same technique used by Rademacher and Zuckerman [4] applied to vector-valued modular forms.

Lemma 2. *Let (F, ρ) be a vector-valued modular form of dimension r and $V = \begin{pmatrix} h' & -\frac{hh'+1}{k} \\ k & -h \end{pmatrix} \in \Gamma$, then if $z = -i(k\tau - h)$, we have*

$$f(e^{-2\pi \frac{z-ih}{k}}) = \Omega_{h,k} \Psi_k(z) f(e^{2\pi i \frac{h'}{k}} e^{-2\pi \frac{1}{kz}}) \tag{14}$$

where $f(x) = (f_1(x), \ldots, f_p(x))^t$,

$$\Psi_k(z) = z^r \begin{pmatrix} e^{2\pi m_1 \frac{z-1/z}{k}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2\pi m_p \frac{z-1/z}{k}} \end{pmatrix} \tag{15}$$

$$\Omega_{h,k} = v^{-1}(V) \left(\rho(V)^{-1}\right)^t \begin{pmatrix} e^{2\pi i m_1 \frac{h'-h}{k}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2\pi i m_p \frac{h'-h}{k}} \end{pmatrix} \tag{16}$$

Proof. First note that if we make $k > 0$ and choose the unique h' such that $0 \leq h' < k$, then $V \in \Gamma$ is determined by h and k, and $\Omega_{h,k}$ only depends on h and k. From the definition of the vector-valued modular (1 and 5), we get that

$$F(\tau) = v(V)^{-1} z^r \rho(V)^{-1} F(V\tau) \tag{17}$$

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_p(x) \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} e^{-2\pi i m_1 \tau} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{-2\pi i m_p \tau} \end{pmatrix} \begin{pmatrix} F_1(\tau) \\ \vdots \\ F_p(\tau) \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} e^{-2\pi i m_1 \tau} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{-2\pi i m_p \tau} \end{pmatrix} v^{-1}(V) z^r \rho(V)^{-1} F(V\tau) \tag{20}$$

$$= v^{-1}(V) z^r \left(\rho(V)^{-1}\right)^t \begin{pmatrix} e^{-2\pi i m_1 \tau} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{-2\pi i m_p \tau} \end{pmatrix} \begin{pmatrix} e^{2\pi i m_1 V\tau} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{2\pi i m_p V\tau} \end{pmatrix} f(e^{2\pi i V\tau}) \tag{21}$$

Now since $-i(k\tau - h) = z$, and since $hh' \equiv -1 \pmod{k}$,

$$\tau = \frac{iz}{k} + \frac{h}{k}, \quad V\tau = \frac{i}{kz} + \frac{h'}{k} \tag{22}$$

Note that since $k > 0$ then $\Re(z) > 0$. Now it only remains to apply (22) into (21) to prove the lemma $\square$

Lemma 3. *Let (F, ρ) be a vector-valued modular form of dimension r, then the Fourier coefficients $a_m = (a_m(1), \ldots, a_m(p))^t$, are given by the following formula*

$$a_m = e^{2\pi N^{-2} m} \sum_{\substack{h,\,k \\ 0 \leq h < k \leq N \\ (h,\,k) = 1}} \Omega_{h,k} e^{-2\pi i h \frac{m}{k}} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} \Psi_k(k(N^{-2} - i\varphi)) \; f\left(e^{\frac{2\pi}{k}\left(ih' - k^{-1}(N^{-2} - i\varphi)^{-1}\right)}\right) e^{-2\pi i m \varphi} d\varphi \tag{23}$$

for any N positive integer, where $\theta'_{h,k}$ and $\theta''_{h,k}$ are given by the Farey dissection of the circle $|x| = e^{2\pi N^{-2}}$, using Farey series of order N.

Proof. Since the functions $f_j(x)$ are analytic inside the unit circle except possibly at zero where there could be a pole, we can use the Cauchy formula to get

$$a_m(j) = \frac{1}{2\pi i}\int_C \frac{f_j(x)}{x^{m+1}}dx \tag{24}$$

where C is the circle $|x| = e^{2\pi N^{-2}}$, for N a positive integer. We can change the path of integration by making the usual dissection of the circle C into arcs $\xi_{h,k}$, using the Farey series of order N. thus we have

$$a_m(j) = \sum_{\substack{h,\,k \\ 0\le h<k\le N \\ (h,\,k)=1}} \frac{1}{2\pi i}\int_{\xi_{h,k}} \frac{f_j(x)}{x^{m+1}}dx \tag{25}$$

We can make the change of variable

$$x = e^{-2\pi N^{-2}+2\pi i\frac{h}{k}+2\pi i\varphi}, \quad -\theta^{'}_{h,k} \le \varphi \le \theta^{''}_{h,k} \tag{26}$$

$$a_m(j) = e^{-2\pi N^{-2}m} \sum_{\substack{h,\,k \\ 0\le h<k\le N \\ (h,\,k)=1}} e^{-2\pi i m\frac{h}{k}} \int_{-\theta^{'}_{h,k}}^{\theta^{''}_{h,k}} f_j(e^{2\pi i\frac{h}{k}-2\pi(N^{-2}-i\varphi)})e^{-2\pi i m\varphi}d\varphi \tag{27}$$

and therefore if we write the same expression in column vectors, we get

$$a_m = e^{-2\pi N^{-2}m} \sum_{\substack{h,\,k \\ 0\le h<k\le N \\ (h,\,k)=1}} e^{-2\pi i m\frac{h}{k}} \int_{-\theta^{'}_{h,k}}^{\theta^{''}_{h,k}} f(e^{2\pi i\frac{h}{k}-2\pi(N^{-2}-i\varphi)})e^{-2\pi i m\varphi}d\varphi \tag{28}$$

now we can apply the result of the previous lemma (14) on the column vector $f(e^{2\pi i\frac{h}{k}-2\pi(N^{-2}-i\varphi)})$, where $z = k(N^{-2}-i\varphi)$ and the lemma is proven. □

Proof of the Main Theorem. We will show now that $f(x)$ in the neighborhood of $x = 0$ is dominated by the principal part $P(x)$, where $P(x)$ is the column vector with components

$$P_j(x) = \sum_{\nu=\mu_j}^{-1} a_\nu(j)x^\nu, \qquad 1 \le j \le p \tag{29}$$

For that purpose we split a_m in two parts

$$a_m = Q_m + R_m \tag{30}$$

where

$$Q_m = e^{2\pi N^{-2}m} \sum_{\substack{h,\,k \\ 0 \le h < k \le N \\ (h,\,k)=1}} \Omega_{h,k} e^{-2\pi i h \frac{m}{k}} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} \Psi_k(k(N^{-2} - i\varphi)) \; P\left(e^{\frac{2\pi}{k}\left(ih' - k^{-1}(N^{-2}-i\varphi)^{-1}\right)}\right) e^{-2\pi i m\varphi} d\varphi \tag{31}$$

$$R_m = e^{2\pi N^{-2}m} \sum_{\substack{h,\,k \\ 0 \le h < k \le N \\ (h,\,k)=1}} \Omega_{h,k} e^{-2\pi i h \frac{m}{k}} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} \Psi_k(k(N^{-2} - i\varphi)) \; D\left(e^{\frac{2\pi}{k}\left(ih' - k^{-1}(N^{-2}-i\varphi)^{-1}\right)}\right) e^{-2\pi i m\varphi} d\varphi \tag{32}$$

to prove the theorem we will show that $R_m = 0$. From the theory of Farey fractions we have

$$\frac{1}{2kN} \le \theta'_{h,k} \le \frac{1}{kN}, \qquad \frac{1}{2kN} \le \theta''_{h,k} \le \frac{1}{kN} \tag{33}$$

and therefore since $k \le N$, we find for $-\theta'_{h,k} \le \varphi \le \theta''_{h,k}$

$$\Re\left(k(N^{-2} - i\varphi)\right) = kN^{-2}, \tag{34}$$

$$\Re\left(\frac{1}{k(N^{-2} - i\varphi)}\right) = \frac{N^{-2}}{k(N^{-4} + \varphi^2)} \ge \frac{N^{-2}}{k(N^{-4} + k^{-2}N^{-2})} = \frac{k}{k^2N^{-2} + 1} \ge \frac{k}{2}, \tag{35}$$

$$|k(N^{-2} - i\varphi)| = k(N^{-4} + \varphi^2)^{\frac{1}{2}} \le (k^2N^{-4} + N^{-2})^{\frac{1}{2}} \le 2^{\frac{1}{2}} N^{-1} \tag{36}$$

$$|e^{\frac{2\pi m_j}{k}\left(k(N^{-2}-i\varphi)) - 1/(k(N^{-2}-i\varphi)\right)}| \le e^{\frac{2\pi m_j}{k}\left(kN^{-2} - \frac{k}{2}\right)} = e^{2\pi m_j N^{-2}} e^{-\pi m_j} \tag{37}$$

Therefore

$$
\begin{aligned}
&|\Psi_k(z_0)| \\
&\quad \leq |(z_0)|^r \sqrt{|e^{\frac{2\pi m_1}{k} z_0 - 1/z_0}|^2 + \ldots + |e^{\frac{2\pi m_p}{k} z_0 - 1/z_0}|^2} \\
&\quad \leq 2^{\frac{r}{2}} N^{-r} \sqrt{p} e^{2\pi m_{max} N^{-2}} e^{-\pi m_{min}}
\end{aligned}
$$

where $z_0 = k(N^{-2} - i\varphi)$, $m_{max} = max\{m_1, \ldots, m_p\}$ and $m_{min} = min\{m_1, \ldots, m_p\}$ Also, using the fact that if $a, b > 0$, then $(a+b)^2 = a^2 + 2ab + b^2 \geq a^2 + b^2$

$$
\begin{aligned}
&| \ D\left(e^{\frac{2\pi}{k}\left(ih' - k^{-1}(N^{-2} - i\varphi)^{-1}\right)}\right) \ | \\
&\quad \leq \sum_{\nu=0}^{\infty} |a_\nu| e^{\frac{-2\pi\nu}{k} \Re\left(k^{-1}(N^{-2} - i\varphi)^{-1}\right)} \\
&\quad \leq \sum_{\nu=0}^{\infty} |a_\nu| e^{-\pi\nu} \\
&\quad = \sum_{\nu=0}^{\infty} \sqrt{|a_\nu(1)|^2 + \ldots + |a_\nu(p)|^2} e^{-\pi\nu} \\
&\quad \leq \sum_{\nu=0}^{\infty} \left(|a_\nu(1)| + \ldots + |a_\nu(p)|\right) e^{-\pi\nu} \\
&\quad = \sum_{\nu=0}^{\infty} |a_\nu(1)| e^{-\pi\nu} + \ldots + \sum_{\nu=0}^{\infty} |a_\nu(p)| e^{-\pi\nu}
\end{aligned}
$$

the last equality holds since all the series converge absolutely because $|e^{-\pi}| \leq$

1

Using these results we have

$$
\begin{aligned}
&|\Psi_k(k(N^{-2} - i\varphi)) D\left(e^{\frac{2\pi}{k}\left(ih' - k^{-1}(N^{-2} - i\varphi)^{-1}\right)}\right) | \\
&\quad \leq 2^{\frac{r}{2}} N^{-r} \sqrt{p} e^{2\pi m_{max} N^{-2}} e^{-\pi m_{min}} \left(\sum_{\nu=0}^{\infty} |a_\nu(1)| e^{-\pi\nu} + \sum_{\nu=0}^{\infty} |a_\nu(2)| e^{-\pi\nu}\right) \\
&\quad = C N^{-r} e^{2\pi m_{max} N^{-2}}
\end{aligned}
$$

where

$$
C = 2^{\frac{r}{2}} \sqrt{p} e^{-\pi m_{min}} \left(\sum_{\nu=0}^{\infty} |a_\nu(1)| e^{-\pi\nu} + \ldots + \sum_{\nu=0}^{\infty} |a_\nu(p)| e^{-\pi\nu}\right) \qquad (38)
$$

Which is finite since $|e^{-\pi}| \leq 1$ and both series are convergent inside the unit circle

Now in order to bound $|\Omega_{h,k}|$, we need to use the Eichler estimate discussed before (12), and therefore we have that

$$
\begin{aligned}
&|\rho(V)| && (39)\\
&\quad \leq K_2 \left(h'^2 + \left(\frac{hh'+1}{k} \right)^2 + k^2 + h^2 \right)^{\alpha} && (40)\\
&\quad \leq K_3 k^{2\alpha} && (41)\\
&\quad \leq K_3 N^{2\alpha} && (42)
\end{aligned}
$$

Where $\alpha = n_1 \log(pK_1)$, and K_2, K_3 are a constants independent of V. Note that (41) holds since in our case we have that $0 \leq h, h' < k$ for all V's

Therefore

$$
\begin{aligned}
&|\Omega_{h,k}|\\
&\quad =| \, v^{-1}(V) \left(\rho(V)^{-1}\right)^t \begin{pmatrix} e^{2\pi i m_1 \frac{h'-h}{k}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2\pi i m_2 \frac{h'-h}{k}} \end{pmatrix} |\\
&\quad \leq K_3 N^{2\alpha} \sqrt{p}\\
&\quad = K_4 N^{2\alpha}
\end{aligned}
$$

thus we have,

$$
\begin{aligned}
&|R_m|\\
&\quad \leq e^{2\pi N^{-2} m} \sum_{\substack{h,\,k \\ 0 \,\leq\, h \,<\, k \,\leq\, N \\ (h,\,k) \,=\, 1}} K_4 N^{2\alpha} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} C N^{-r} e^{2\pi\alpha N^{-2}} d\varphi\\
&\quad \leq K_5 e^{2\pi N^{-2}(m+\alpha)} N^{-r+2\alpha} \sum_{\substack{h,\,k \\ 0 \,\leq\, h \,<\, k \,\leq\, N \\ (h,\,k) \,=\, 1}} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} d\varphi\\
&\quad = K_5 e^{2\pi N^{-2}(m+\alpha)} N^{-r+2\alpha}
\end{aligned}
$$

where K_5 is a constant independent of V, and since N can be very large, we can conclude that $|R_m| = 0$ for $r > 2\alpha$. $\square$

References

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