Construction of Vector Valued Modular Forms of positive dimension Part II

Pepe Gimenez

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1 Construction of a vector-valued modular form of positive dimension r

So far we have that for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$

$$\varepsilon^{-1}(M)\rho^{-1}(M)\left(-i(\gamma\tau+\delta)\right)^{r} F_{\nu}(M\tau) - F_{\nu}(\tau) = P_{M,\nu}(\tau, r, \varepsilon, \rho) \tag{1}$$

where $P_{M,\nu}(\tau, r, \varepsilon, \rho)$ is a column vector of polynomials of degree at most r. which automatically implies that

$$\varepsilon^{-1}(M)\rho^{-1}(M)\left(-i(\gamma\tau+\delta)\right)^{r} F(M\tau) - F(\tau) = Q_{M}(\tau, r, \varepsilon, \rho) \tag{2}$$

where $Q_M(\tau, r, \varepsilon, \rho)$ is a column vector of polynomials of degree at most r, since

$$F(\tau) = \begin{pmatrix} \sum_{\nu=1}^{\mu} b_{\nu}(1)e^{2\pi i(m_1-\nu)\tau} + \sum_{m=0}^{\infty} a_m(1)e^{2\pi i(m+m_1)\tau} \\ \vdots \\ \sum_{\nu=1}^{\mu} b_{\nu}(p)e^{2\pi i(m_p-\nu)\tau} + \sum_{m=0}^{\infty} a_m(p)e^{2\pi i(m+m_p)\tau} \end{pmatrix} = \begin{pmatrix} \sum_{\nu=1}^{\mu} b_{\nu}(1)F_{\nu}(\tau)(1) \\ \vdots \\ \sum_{\nu=1}^{\mu} b_{\nu}(p)F_{\nu}(\tau)(p) \end{pmatrix}, \tag{3}$$

now let $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By the definition of $F(\tau)$ it is easy to see that for $n \in \mathbb{Z}$ we have

$$F(S^n\tau) = \begin{pmatrix} e^{2\pi i n m_1} & & \\ & \ddots & \\ & & e^{2\pi i n m_p} \end{pmatrix} F(\tau) = \varepsilon(S^n)\rho(S^n)\left(-i\right)^{r} F(\tau) \tag{4}$$

and therefore $Q_{S^n}(\tau, r, \varepsilon, \rho) \equiv 0$, and since all the elements of $\Gamma(1)$ can be written as a product of S^n and T for $n \in \mathbb{Z}$, now we want to find $b_1, \ldots, b_\mu$ such that $\varepsilon^{-1}(T)\rho^{-1}(T)\left(-i\tau\right)^r F(T\tau) - F(\tau) = 0$, or what is the same

$$\varepsilon^{-1}(T)\rho^{-1}(T)\left(-i\tau\right)^r F(T\tau) - F(\tau) = \begin{pmatrix} \sum_{\nu=1}^{\mu} b_\nu(1) P_{T,\nu}(\tau, r, \varepsilon, \rho)(1) \\ \vdots \\ \sum_{\nu=1}^{\mu} b_\nu(p) P_{T,\nu}(\tau, r, \varepsilon, \rho)(p) \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} Q_T(\tau, r, \varepsilon, \rho)(1) \\ \vdots \\ Q_T(\tau, r, \varepsilon, \rho)(p) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{6}$$

because if we do so we will have a function $F(\tau)$ with the transformation law

$$F(\tau) = \varepsilon^{-1}(M)\rho^{-1}(M)\left(-i(\gamma\tau + \delta)\right)^r F(M\tau) \tag{7}$$

and since it is regular in $\mathcal{H}$, and has the expansion at ∞ (3), then $F(\tau)$ is a modular form of dimension r.

Now if we replace τ by $T\tau$ in (5), we see that

$$\varepsilon^{-1}(T)\rho^{-1}(T)\left(-i\tau\right)^r \begin{pmatrix} Q_T(\frac{-1}{\tau}, r, \varepsilon, \rho)(1) \\ \vdots \\ Q_T(\frac{-1}{\tau}, r, \varepsilon, \rho)(p) \end{pmatrix} = \begin{pmatrix} -Q_T(\tau, r, \varepsilon, \rho)(1) \\ \vdots \\ -Q_T(\tau, r, \varepsilon, \rho)(p) \end{pmatrix} \tag{8}$$

and if $Q_T(\tau_o, r, \varepsilon, \rho) = 0$ then we have an homogeneous system of p equations in the p unknowns $Q_T(\frac{-1}{\tau_o}, r, \varepsilon, \rho)(1), \ldots, Q_T(\frac{-1}{\tau_o}, r, \varepsilon, \rho)(p)$, and since $det(\rho^{-1}(T)) \neq 0$, we have that $Q_T(\frac{-1}{\tau_o}, r, \varepsilon, \rho)(j) = 0$ for $1 \leq j \leq p$. Therefore the zeros of $Q_T(\tau, r, \varepsilon, \rho)$ occur in pairs except for $\tau = \pm i$ now suppose that exists a set $\tau_1, \ldots, \tau_n$ of distinct roots of $Q_T(\tau, r, \varepsilon, \rho)$, such that $n = [r/2] + 1$, also that $-1/\tau_j$ is not included in that set for any $1 \leq j \leq n$ and neither does $\pm i$, then since $Q_T(\tau_j, r, \varepsilon, \rho) = 0$ for all the elements in the set, the number of zeros of each of the p polynomials will be $2([r/2] + 1) > r$, and therefore $Q_T(\tau, r, \varepsilon, \rho) \equiv 0$.

Now consider the homogeneous linear system corresponding to the k^{th} element in $Q_T(\tau_j, r, \varepsilon, \rho)$

$$\sum_{\nu=1}^{\mu} b_\nu(k) P_{T,\nu}(\tau_j, r, \varepsilon, \rho)(k) = 0 \tag{9}$$

of $2([r/2]+1)$ equations in the μ unknowns $b_1(k), \ldots, b_\mu(k)$, and if μ is greater than $2([r/2]+1)$, we can find solutions for $b_1(k), \ldots, b_\mu(k)$. Therefore we can state the following theorem

Theorem 1. *Let μ be an integer greater than $2([r/2]+1)$. If we define $F(\tau)$ as in (3) with $r > 2\alpha$ and $b_1, \ldots, b_\mu$ being column vectors of dimension p such that the k^{th} element satisfies (9) for each $1 \le k \le p$ then $F(\tau)$ is a vector-valued modular form of dimension r*

2 The supplementary series

Let m'_s and ν' be defined by

$$\begin{array}{ccc} m'_s = 1 - m_s & \nu' = 1 - \nu & \text{if } m_s > 0 \\ m'_s = 0 & \nu' = -\nu & \text{if } m_s = 0 \end{array} \tag{10}$$

Further we can define

$$\varepsilon'(V) = e^{i\pi r}\varepsilon^{-1}(V), \quad \text{and} \quad \rho'(V) = \overline{\rho(V)} \tag{11}$$

since r is an integer and ε is a multiplier system for Γ, it follows that ε' is also a multiplier system for Γ. On the other hand since ρ is a representation for Γ, we have that ρ' is also a representation. Now let $a_m(\nu', r, \varepsilon', \rho')$ be defined as in (2, Part I), therefore

$$a_m(\nu', r, \varepsilon', \rho') = 2\pi \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{d,\, c \\ 0 \le d < c \\ (d,\, c) = 1}} e^{-2\pi i \frac{dm + d'\nu'}{c}} \varepsilon'^{-1}(V_{c,d})\rho'^{-1}(V_{c,d}) \begin{pmatrix} e^{2\pi i m'_1 \frac{d'-d}{c}} B^1_{c,\nu',m,\rho',\varepsilon',r} \\ \vdots \\ e^{2\pi i m'_p \frac{d'-d}{c}} B^1_{c,\nu',m,\rho',\varepsilon',r} \end{pmatrix} \tag{12}$$

now we define the supplementary series $\widehat{F}_{\nu'}(\tau)$ of the series $F_\nu(\tau)$ by

$$\widehat{F}_{\nu'}(\tau) = \begin{pmatrix} e^{2\pi i (m'_1 - \nu')\tau} \\ \vdots \\ e^{2\pi i (m'_p - \nu')\tau} \end{pmatrix} + \begin{pmatrix} \sum_{m=0}^{\infty} a_m(\nu', r, \varepsilon', \rho')(1) e^{2\pi i (m + m'_1)\tau} \\ \vdots \\ \sum_{m=0}^{\infty} a_m(\nu', r, \varepsilon', \rho')(p) e^{2\pi i (m + m'_p)\tau} \end{pmatrix} \tag{13}$$

we can see using exactly the same arguments as before that

1. $\widehat{F}_{\nu'}(\tau)$ is regular for $\tau \in \mathcal{H}$

2. $\widehat{F}_{\nu'}(\tau) - \varepsilon'^{-1}(M)\rho'^{-1}(M)\left(-i(\gamma\tau+\delta)\right)^r \widehat{F}_{\nu'}(M\tau) = \widehat{P}_{M,\nu'}(\tau, r, \varepsilon', \rho')$, for every $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$

3. The column vector of polynomials $\widehat{P}_{M,\nu'}(\tau, r, \varepsilon', \rho')$ is given by the following formula

$$\widehat{P}_{M,\nu'}(\tau, r, \varepsilon', \rho') \quad = \quad \widehat{H}_{M,\nu'}(\tau, r, \varepsilon', \rho') + \widehat{K}_{\nu',r,\varepsilon',\rho'}(\tau, M) \tag{14}$$

where $\widehat{K}_{\nu',r,\varepsilon',\rho'}(\tau, M)$ is given by (50 part I)

$$\widehat{K}_{\nu',r,\varepsilon',\rho'}(\tau, M) = \left(I - \left(-i(\gamma\tau+\delta)\right)^r \varepsilon'^{-1}(M)\rho'^{-1}(M)\right)\frac{1}{2}a_0^*(\nu', r, \varepsilon', \rho') \tag{15}$$

where

$$a_0^*(\nu', r, \varepsilon', \rho') = 2\pi \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{d,\, c \\ 0 \,\le\, d \,<\, c \\ (d,\, c) \,=\, 1}} \varepsilon'^{-1}(V_{c,d})\rho'^{-1}(V_{c,d}) \begin{pmatrix} e^{2\pi i \frac{d'(m'_1-\nu')}{c}} \left(\frac{2\pi\nu'}{c}\right)^{r+1} \frac{1}{(r+1)!}[1-m'_1] \\ \vdots \\ e^{2\pi i \frac{d'(m'_p-\nu')}{c}} \left(\frac{2\pi\nu'}{c}\right)^{r+1} \frac{1}{(r+1)!}[1-m'_p] \end{pmatrix} \tag{16}$$

here $[x]$ is the greatest integer less than or equal to x. And $\widehat{H}_{M,\nu'}(\tau, r, \varepsilon', \rho')$ is given by (104 part I)

$$\widehat{H}_{M,\nu'}(\tau, r, \varepsilon', \rho') \tag{17}$$

$$= \widehat{G}_{\nu'}(\tau) - \varepsilon'^{-1}(M)\rho'^{-1}(M)\left(-i(\gamma\tau+\delta)\right)^r \widehat{G}_{\nu'}(M\tau) \tag{18}$$

$$= \widehat{F}_{\nu'}(\tau) - \varepsilon'^{-1}(M)\rho'^{-1}(M)\left(-i(\gamma\tau+\delta)\right)^r \widehat{F}_{\nu'}(M\tau) - \widehat{K}_{\nu',r,\varepsilon',\rho'}(\tau, M) \tag{19}$$

$$= \lim_{N\to\infty}\Big(\sum_{c=1}^{\infty} \sum_{\substack{(c,\, d) \,\in\, \mathcal{J}_M(N) \\ d \,\in\, D^c}} \varepsilon'^{-1}(V_{c,d}M)\rho'^{-1}(V_{c,d}M)\left(-i(\gamma\tau+\delta)\right)^r \tag{20}$$

$$\left(-i(cM\tau+d)\right)^r \begin{pmatrix} e^{2\pi i \frac{d'(m'_1-\nu')}{c}} \sum_{p=0}^{r} \frac{1}{p!} \left(\frac{2\pi i(\nu'-m'_1)}{c(cM\tau+d)}\right)^p \\ \vdots \\ e^{2\pi i \frac{d'(m'_p-\nu')}{c}} \sum_{p=0}^{r} \frac{1}{p!} \left(\frac{2\pi i(\nu'-m'_p)}{c(cM\tau+d)}\right)^p \end{pmatrix} \tag{21}$$

$$- \sum_{\substack{c \,\in\, \mathbb{Z} \\ 0 \,<\, c \,\le\, tN}} \sum_{\substack{d \,\in\, D^c \\ |d| \,\le\, N}} \varepsilon'^{-1}(V_{c,d})\rho'^{-1}(V_{c,d})\left(-i(c\tau+d)\right)^r \tag{22}$$

$$\begin{pmatrix} e^{2\pi i \frac{d'(m'_1-\nu')}{c}} \sum_{p=0}^{r} \frac{1}{p!} \left(\frac{2\pi i(\nu'-m'_1)}{c(c\tau+d)}\right)^p \\ \vdots \\ e^{2\pi i \frac{d'(m'_p-\nu')}{c}} \sum_{p=0}^{r} \frac{1}{p!} \left(\frac{2\pi i(\nu'-m'_p)}{c(c\tau+d)}\right)^p \end{pmatrix}\Big) \tag{23}$$

It is easy to see from the formulas that for all $M \in \Gamma$ we have

1. $\widehat{H}_{M,\nu'}(\tau, r, \varepsilon', \rho') = \overline{H_{M,\nu}(\overline{\tau}, r, \varepsilon, \rho)}$
2. $a_0^*(\nu', r, \varepsilon', \rho') = -\overline{a_0^*(\nu, r, \varepsilon, \rho)}$
3. $\widehat{K}_{\nu', r, \varepsilon', \rho'}(\tau, M) = -\overline{K_{\nu, r, \varepsilon, \rho}(\overline{\tau}, M)}$
4. $\widehat{P}_{M,\nu'}(\tau, r, \varepsilon', \rho') = \overline{P_{M,\nu}(\overline{\tau}, r, \varepsilon, \rho)} - 2\overline{K_{\nu, r, \varepsilon, \rho}(\overline{\tau}, M)}$

Now let $b_1, \ldots, b_\mu$ be a set of column vectors such that $b_j \in \mathbb{C}^p$, $b_\mu \neq 0$, $\rho : \Gamma \longrightarrow GL(p, \mathbb{C})$ a p-dimensional complex representation , ε a Multiplier System , and $r > 2\alpha$, $r, \mu \in \mathbb{Z}^+$ and let $F(\tau)$ be defined as in (41 part I)

$$F(\tau) = \begin{pmatrix} \sum_{\nu=1}^{\mu} b_\nu(1) F_\nu(\tau)(1) \\ \vdots \\ \sum_{\nu=1}^{\mu} b_\nu(p) F_\nu(\tau)(p) \end{pmatrix} \tag{24}$$

then we can define the supplementary series $\widehat{F}(\tau)$ of $F(\tau)$ by

$$\widehat{F}(\tau) = \begin{pmatrix} \sum_{\nu'=-\mu}^{-1} \overline{b}_{\nu'}(1) \widehat{F}_{\nu'}(\tau)(1) \\ \vdots \\ \sum_{\nu'=-\mu}^{-1} \overline{b}_{\nu'}(p) \widehat{F}_{\nu'}(\tau)(p) \end{pmatrix} \tag{25}$$

For $M \in \Gamma(1)$ we have that

$$\varepsilon^{-1}(M)\rho^{-1}(M)\left(-i(\gamma\tau + \delta)\right)^r F(M\tau) - F(\tau) = Q_M(\tau, r, \varepsilon, \rho) \tag{26}$$

where

$$Q_M(\tau, r, \varepsilon, \rho) = \begin{pmatrix} \sum_{\nu=1}^{\mu} b_\nu(1) P_{M,\nu}(\tau, r, \varepsilon, \rho)(1) \\ \vdots \\ \sum_{\nu=1}^{\mu} b_\nu(p) P_{M,\nu}(\tau, r, \varepsilon, \rho)(p) \end{pmatrix} \tag{27}$$

and

$$\varepsilon'^{-1}(M)\rho'^{-1}(M)\left(-i(\gamma\tau + \delta)\right)^r \widehat{F}(M\tau) - \widehat{F}(\tau) = \widehat{Q}_M(\tau, r, \varepsilon', \rho') \tag{28}$$

where

$$\widehat{Q}_M(\tau, r, \varepsilon', \rho') = \begin{pmatrix} \sum_{\nu=-\mu}^{-1} \overline{b}_{\nu'}(1) \widehat{P}_{M,\nu'}(\tau, r, \varepsilon', \rho')(1) \\ \vdots \\ \sum_{\nu=-\mu}^{-1} \overline{b}_{\nu'}(p) \widehat{P}_{M,\nu'}(\tau, r, \varepsilon', \rho')(p) \end{pmatrix} \tag{29}$$

Theorem 2.

$$F(\tau) \in \{\Gamma(1), r, \varepsilon, \rho\} \Leftrightarrow \widehat{F}(\tau) = -\begin{pmatrix} 2\sum_{\nu=1}^{\mu} \overline{b_\nu}(1)\overline{K_{\nu,r,\varepsilon,\rho}(\overline{\tau}, T)(1)} \\ \vdots \\ 2\sum_{\nu=1}^{\mu} \overline{b_\nu}(p)\overline{K_{\nu,r,\varepsilon,\rho}(\overline{\tau}, T)(p)} \end{pmatrix} \tag{30}$$

Proof. We saw that $F(\tau) \in \{\Gamma(1), r, \varepsilon, \rho\}$ if and only if $Q_T(\tau, r, \varepsilon, \rho) = 0$.

$$\widehat{Q}_T(\tau, r, \varepsilon', \rho') \tag{31}$$

$$= \begin{pmatrix} \sum_{\nu=1}^{\mu} \overline{b_\nu}(1)\widehat{P}_{T,\nu'}(\tau, r, \varepsilon', \rho')(1) \\ \vdots \\ \sum_{\nu=1}^{\mu} \overline{b_\nu}(p)\widehat{P}_{T,\nu'}(\tau, r, \varepsilon', \rho')(p) \end{pmatrix} \tag{32}$$

$$= \begin{pmatrix} \sum_{\nu=1}^{\mu} \overline{b_\nu}(1)\left(\overline{P_{T,\nu}(\overline{\tau}, r, \varepsilon, \rho)}(1) - 2\overline{K_{\nu,r,\varepsilon,\rho}(\overline{\tau}, T)}(1)\right) \\ \vdots \\ \sum_{\nu=1}^{\mu} \overline{b_\nu}(p)\left(\overline{P_{T,\nu}(\overline{\tau}, r, \varepsilon, \rho)}(p) - 2\overline{K_{\nu,r,\varepsilon,\rho}(\overline{\tau}, T)}(p)\right) \end{pmatrix} \tag{33}$$

$$= \overline{Q_T(\overline{\tau}, r, \varepsilon, \rho)} - \begin{pmatrix} 2\sum_{\nu=1}^{\mu} \overline{b_\nu}(1)\overline{K_{\nu,r,\varepsilon,\rho}(\overline{\tau}, T)(1)} \\ \vdots \\ 2\sum_{\nu=1}^{\mu} \overline{b_\nu}(p)\overline{K_{\nu,r,\varepsilon,\rho}(\overline{\tau}, T)(p)} \end{pmatrix} \tag{34}$$

On the other hand we can see that since $\widehat{F}_{\nu'}(\tau)$ is regular in $\mathcal{H}$, so is $\widehat{F}(\tau)$, also by (13) we can see that each $\widehat{F}_{\nu'}(\tau)$ is bounded at ∞, and therefore so is $\widehat{F}(\tau)$ □

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