Construction of Vector Valued Modular Forms of positive dimension

Pepe Gimenez

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Theorem 1. *Let $b_1, \ldots, b_\mu$ be a set of column vectors such that $b_j \in \mathbb{C}^p$, $b_\mu \neq 0$, $\rho : \Gamma \longrightarrow GL(p, \mathbb{C})$ a p-dimensional complex representation , ε a Multiplier System , and $r > 2\alpha$ (**??**), $r, \mu \in \mathbb{Z}^+$ then if*

$$F(\tau) = \begin{pmatrix} \sum_{\nu=1}^{\mu} b_\nu(1) e^{2\pi i(m_1-\nu)\tau} + \sum_{m=0}^{\infty} a_m(1) e^{2\pi i(m+m_1)\tau} \\ \vdots \\ \sum_{\nu=1}^{\mu} b_\nu(p) e^{2\pi i(m_p-\nu)\tau} + \sum_{m=0}^{\infty} a_m(p) e^{2\pi i(m+m_p)\tau} \end{pmatrix}, \tag{1}$$

where $a_m(\nu, r, \varepsilon)$ is a column vector in which the j^{th} component is given by

$$a_m(\nu, r, \varepsilon)(j) = 2\pi \sum_{c=1}^{\infty} \frac{1}{c} \left[\sum_{\substack{d,\, c \\ 0 \leq d < c \\ (d,\, c) = 1}} \varepsilon^{-1}(V_{c,d}) \left(\sum_{s=1}^{p} x_{js}(c, d) e^{2\pi i m_s \frac{d'-d}{c}} B^s_{c,\nu,m,\rho,\varepsilon,r} \right) e^{-2\pi i \frac{dm+d'\nu}{c}} \right] \tag{2}$$

where

$$\rho^{-1}(V_{c,d}) = \begin{pmatrix} x_{11}(c,d) & \cdots & x_{1p}(c,d) \\ \vdots & \ddots & \vdots \\ x_{p1}(c,d) & \cdots & x_{pp}(c,d) \end{pmatrix} \tag{3}$$

where

$$V_{c,d} = \begin{pmatrix} d' & -\frac{dd'+1}{c} \\ c & d \end{pmatrix} \in \Gamma(1) \tag{4}$$

$$B^s_{c,\nu,m,\rho,\varepsilon,r} \tag{5}$$

$$= \left(\frac{\nu - m_s}{m + m_s}\right)^{\frac{r+1}{2}} I_{r+1}\left(\frac{4\pi}{c}(\nu - m_s)^{\frac{1}{2}}(m + m_s)^{\frac{1}{2}}\right) \quad if \quad m + m_s > 0 \tag{6}$$

$$= \frac{2\pi}{(r+1)!}\left(\frac{2\pi\nu}{c}\right)^{r+1} \quad if \quad m = m_s = 0 \tag{7}$$

$$a_m = \begin{pmatrix} \sum_{\nu=1}^{\mu} b_\nu(1) a_m(\nu, r, \varepsilon)(1) \\ \vdots \\ \sum_{\nu=1}^{\mu} b_\nu(p) a_m(\nu, r, \varepsilon)(p) \end{pmatrix} = \begin{pmatrix} a_m(1) \\ \vdots \\ a_m(p) \end{pmatrix} \tag{8}$$

then

1. *$F(\tau)$ is regular in the complex upper half-plane $\mathcal{H}$*

2. *$F(\tau)$ satisfies*

$$F(\tau) - \varepsilon^{-1}(M)(-i(c\tau + d))^r \rho^{-1}(M) F(M\tau) = p_M(\tau) \tag{9}$$

, for all

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \tag{10}$$

where $p_M(\tau)$ is a column vector of polynomials on τ of degree at most r.

Lemma 2. *For $r > 2\alpha$ as $m \to \infty$, we have*

$$|a_m(\nu, r, \varepsilon)| = O\left((m + \kappa_m)^{-\frac{3}{4} - \frac{r}{2}} e^{4\pi(\nu - \kappa_m)^{\frac{1}{2}}(m + \kappa_M)^{\frac{1}{2}}}\right) \tag{11}$$

where

$$\kappa_m = \min m_1, \ldots, m_p \quad and \quad \kappa_M = \max m_1, \ldots, m_p \tag{12}$$

Proof. In [1] Marvin Knopp proves that given $0 \le m_s < 1$ and $q \ge 0$

$$c^{-q} A_{c,\nu,m_s}(m) = O\left(c^{\frac{2}{3} + \varepsilon}\right) \tag{13}$$

where

$$A_{c,\nu,m_s}(m) = \sum_{\substack{d,\, c \\ 0 \le d < c \\ (d,\, c) = 1}} \varepsilon^{-1}(V_{c,d}) e^{2\pi i m_s \frac{d' - d}{c}} e^{-2\pi i \frac{dm + d'\nu}{c}} \tag{14}$$

the strategy is the same as the one in the proof of Knopp first we can fix s and show that

$$2\pi\sum_{c=2}^{\infty}\frac{1}{c}\left[\sum_{\substack{d,\,c\\0\le d<c\\(d,\,c)=1}}\varepsilon^{-1}(V_{c,d})\left(x_{sj}(c,d)e^{2\pi i m_s\frac{d'-d}{c}}B^s_{c,\nu,m,r}\right)e^{-2\pi i\frac{dm+d'\nu}{c}}\right] \le C(m+m_s)^{-\frac{1}{2}}e^{2\pi(m+m_s)^{\frac{1}{2}}(\nu-m_s)^{\frac{1}{2}}} \tag{15}$$

and show that the summations behaves as the first term as $m\to\infty$. Here $B^s_{c,\nu,m,r}$ can be understood as $B^s_{c,\nu,m,\rho,\varepsilon,r}$, with a MS ε and a representation ρ such that

$$\varepsilon\begin{pmatrix}1&1\\0&1\end{pmatrix}\rho\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}e^{2\pi i m_1}&&s^{th}cloumn&&\\&\ddots&&&\\s^{th}row&&e^{2\pi i m_s}&&\\&&&\ddots&\\&&&&e^{2\pi i m_p}\end{pmatrix} \tag{16}$$

Also in the previous paper I showed that $|\rho(V_{c,d})|=O(c^{2\alpha})$, which means that $|x_{ij}|=O(c^{2\alpha})$. And since $r>2\alpha$, we can write $r=q+\alpha$ where $q>0$

In the same fashion as the mentioned proof we can see that

$$2\pi\sum_{c=2}^{\infty}\frac{1}{c}\sum_{\substack{d,\,c\\0\le d<c\\(d,\,c)=1}}\left|\varepsilon^{-1}(V_{c,d})x_{sj}(c,d)e^{2\pi i m_s\frac{d'-d}{c}}B^s_{c,\nu,m,r}e^{-2\pi i\frac{dm+d'\nu}{c}}\right| \tag{17}$$

$$\le C_1\frac{(\nu-m_s)^{r+\frac{1}{2}}}{(m+m_s)^{\frac{1}{2}}}\sinh\left(2\pi(m+m_s)^{\frac{1}{2}}(\nu-m_s)^{\frac{1}{2}}\right)\sum_{c=2}^{\infty}x_{sj}(c,d)c^{-r-2}A_{c,\nu,m_s} \tag{18}$$

$$\le C_2(m+m_s)^{-\frac{1}{2}}e^{2\pi(m+m_s)^{\frac{1}{2}}(\nu-m_s)^{\frac{1}{2}}}\sum_{c=2}^{\infty}c^{-\frac{4}{3}+\varepsilon} \tag{19}$$

$$\le C_3(m+m_s)^{-\frac{1}{2}}e^{2\pi(m+m_s)^{\frac{1}{2}}(\nu-m_s)^{\frac{1}{2}}} \tag{20}$$

the first term is

$$2\pi\varepsilon^{-1}(V_{1,0})x_{1,s}(1,0)e^{2\pi i m_s(d'-d)}e^{-2\pi i(dm+d'\nu)}\left(\frac{\nu-m_s}{m+m_s}\right)^{\frac{r+1}{2}}I_{r+1}\left(4\pi(\nu-m_s)^{\frac{1}{2}}(m+m_s)^{\frac{1}{2}}\right) = O\left((m+m_s)^{-\frac{r}{2}-\frac{3}{4}}e^{4\pi(m+m_s)^{\frac{1}{2}}(\nu-m_s)^{\frac{1}{2}}}\right) \tag{21}$$

therefore we can see that the series

$$2\pi \sum_{c=2}^{\infty} \frac{1}{c} A_{c,\nu,m_s}(m) B^s_{c,\nu,m,r} \tag{22}$$

converges uniformly on compacts and absolutely. So can conclude that

$$\begin{aligned}
a_m(\nu, r, \varepsilon)(j) &= 2\pi \sum_{c=1}^{\infty} \frac{1}{c} \left[\sum_{\substack{d,\, c \\ 0 \le d < c \\ (d,\, c) = 1}} \varepsilon^{-1}(V_{c,d}) \left(\sum_{s=1}^{p} x_{sj}(c,d) e^{2\pi i m_s \frac{d'-d}{c}} B^s_{c,\nu,m,\rho,\varepsilon,r} \right) e^{-2\pi i \frac{dm+d'\nu}{c}} \right] && (23) \\
&\le 2\pi \sum_{s=1}^{p} \sum_{c=1}^{\infty} \frac{1}{c} \left[\sum_{\substack{d,\, c \\ 0 \le d < c \\ (d,\, c) = 1}} \left| \varepsilon^{-1}(V_{c,d}) \left(x_{sj}(c,d) e^{2\pi i m_s \frac{d'-d}{c}} B^s_{c,\nu,m,\rho,\varepsilon,r} \right) e^{-2\pi i \frac{dm+d'\nu}{c}} \right| \right] && (24) \\
&= O\left((m+\kappa_m)^{-\frac{3}{4}-\frac{r}{2}} e^{4\pi(\nu-\kappa_m)^{\frac{1}{2}}(m+\kappa_M)^{\frac{1}{2}}} \right) && (25)
\end{aligned}$$

□

Corollary 3. *The following series converges uniformly on* $I_w = \{\tau : \mathcal{I}(\tau) > w > 0\}$

$$\sum_{m=0}^{\infty} \left| a_m(\nu, r, \varepsilon)(j) e^{2\pi i (m+m_j)\tau} \right| \tag{26}$$

Proof.

$$\begin{aligned}
&\left| a_m(\nu, r, \varepsilon)(j) e^{2\pi i (m+m_j)\tau} \right| && (27) \\
&\quad \le C_1 \left| (m+\kappa_m)^{-\frac{3}{4}-\frac{r}{2}} e^{4\pi(\nu-\kappa_m)^{\frac{1}{2}}(m+\kappa_M)^{\frac{1}{2}}} e^{2\pi i (m+m_j)\tau} \right| && (28) \\
&\quad \le C_2 m^{-\frac{3}{4}-\frac{r}{2}} e^{-2\pi m w + 4\pi \mu^{\frac{1}{2}}(m+1)^{\frac{1}{2}}} && (29) \\
&\quad \le C_3 m^{-\frac{3}{4}} \quad \text{for } m \text{ large enough} && (30)
\end{aligned}$$

□

Proof of the Theorem. Let $F_\nu(\tau)$ be a column function defined in the following way

$$F_\nu(\tau) \tag{31}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} + \sum_{m=0}^{\infty} \begin{pmatrix} a_m(\nu,r,\varepsilon)(1)e^{2\pi i(m+m_1)\tau} \\ \vdots \\ a_m(\nu,r,\varepsilon)(p)e^{2\pi i(m+m_p)\tau} \end{pmatrix} \tag{32}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} + \sum_{m=0}^{\infty} \begin{pmatrix} e^{2\pi i(m+m_1)\tau} & & \\ & \ddots & \\ & & e^{2\pi i(m+m_p)\tau} \end{pmatrix} \begin{pmatrix} a_m(\nu,r,\varepsilon)(1) \\ \vdots \\ a_m(\nu,r,\varepsilon)(p) \end{pmatrix} \tag{33}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} + 2\pi \sum_{m=0}^{\infty} \begin{pmatrix} e^{2\pi i(m+m_1)\tau} & & \\ & \ddots & \\ & & e^{2\pi i(m+m_p)\tau} \end{pmatrix} \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{d,\,c \\ 0 \le d < c \\ (d,\,c)=1}} \varepsilon^{-1}(V_{c,d}) \tag{34}$$

$$\left(\rho^{-1}(V_{c,d})\right)^t \begin{pmatrix} e^{2\pi i m_1 \frac{d'-d}{c}} & & \\ & \ddots & \\ & & e^{2\pi i m_p \frac{d'-d}{c}} \end{pmatrix} \begin{pmatrix} e^{-2\pi i \frac{dm+d'\nu}{c}} B^1_{c,\nu,m,\rho,\varepsilon,r} \\ \vdots \\ e^{-2\pi i \frac{dm+d'\nu}{c}} B^p_{c,\nu,m,\rho,\varepsilon,r} \end{pmatrix} \tag{35}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} + 2\pi \sum_{m=0}^{\infty} \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{d,\,c \\ 0 \le d < c \\ (d,\,c)=1}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d}) \begin{pmatrix} e^{2\pi i m_1 \frac{d'-d}{c}} & & \\ & \ddots & \\ & & e^{2\pi i m_p \frac{d'-d}{c}} \end{pmatrix} \tag{36}$$

$$\begin{pmatrix} e^{2\pi i(m+m_1)\tau} & & \\ & \ddots & \\ & & e^{2\pi i(m+m_p)\tau} \end{pmatrix} \begin{pmatrix} e^{-2\pi i \frac{dm+d'\nu}{c}} B^1_{c,\nu,m,\rho,\varepsilon,r} \\ \vdots \\ e^{-2\pi i \frac{dm+d'\nu}{c}} B^p_{c,\nu,m,\rho,\varepsilon,r} \end{pmatrix} \tag{37}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} + 2\pi \sum_{m=0}^{\infty} \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{d,\,c \\ 0 \le d < c \\ (d,\,c)=1}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d}) \tag{38}$$

$$\begin{pmatrix} e^{2\pi i m_1 \frac{d'-d}{c}} & & \\ & \ddots & \\ & & e^{2\pi i m_p \frac{d'-d}{c}} \end{pmatrix} \begin{pmatrix} e^{-2\pi i \frac{dm+d'\nu}{c}} B^1_{c,\nu,m,\rho,\varepsilon,r} e^{2\pi i(m+m_1)\tau} \\ \vdots \\ e^{-2\pi i \frac{dm+d'\nu}{c}} B^p_{c,\nu,m,\rho,\varepsilon,r} e^{2\pi i(m+m_p)\tau} \end{pmatrix} \tag{39}$$

$$\tag{40}$$

By the previous corollary and lemma we have that $F(\tau)(j)$ converges absolutely on m and on c, and therefore the above interchanges are justified. Also we can see that $F(\tau)$ can be written in the following form

$$F(\tau) = \begin{pmatrix} \sum_{\nu=1}^{\mu} b_\nu(1) F_\nu(\tau)(1) \\ \vdots \\ \sum_{\nu=1}^{\mu} b_\nu(p) F_\nu(\tau)(p) \end{pmatrix} \tag{41}$$

also since the series converges, $F(\tau)$ is regular in $\mathcal{H}$. We will prove the result for $\tau = iy$ and $y > 0$, and by analytic continuation the result will follow for τ on $\mathcal{H}$.

Now let us rewrite the function $F_\nu(\tau)(j)$ in the following manner

$$\begin{aligned} F_\nu(\tau)(j) - e^{2\pi i(m_j-\nu)\tau} \\ = 2\pi \sum_{m=0}^{\infty} \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{d,\, c \\ 0 \le d < c \\ (d,\, c) = 1}} \varepsilon^{-1}(V_{c,d}) \left(\sum_{s=1}^{p} x_{js}(c,d) e^{2\pi i m_s \frac{d'-d}{c}} \right) \\ e^{-2\pi i \frac{dm+d'\nu}{c}} B^s_{c,\nu,m,\rho,\varepsilon,r} e^{2\pi i(m+m_s)\tau} \\ = 2\pi \sum_{m=0}^{\infty} \sum_{s=1}^{p} \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{d,\, c \\ 0 \le d < c \\ (d,\, c) = 1}} \varepsilon^{-1}(V_{c,d}) x_{js}(c,d) e^{2\pi i \frac{d'(m_s-\nu)}{c}} \\ B^s_{c,\nu,m,\rho,\varepsilon,r} e^{2\pi i(m+m_s)(\tau-\frac{d}{c})} \end{aligned}$$

to continue with the proof we will need the Lipschitz summation formula for $p > -1$, $0 \le m_s < 1$ and $\mathcal{R}(t) > 0$

$$\sum_{n=0}^{\infty} (n+m_s)^p e^{2\pi i t(n+m_s)} \tag{42}$$

$$= \frac{\Gamma(p+1)}{(2\pi)^{p+1}} \sum_{q=-\infty}^{\infty} e^{2\pi i m_s} (t-qi)^{-p-1}, \quad \text{if } m_s + p > 0 \tag{43}$$

$$= -\frac{1}{2} + \frac{1}{2\pi} \lim_{N\to\infty} \sum_{q=-N} N(t+qi)^{-1}, \quad \text{if } m_s = p = 0 \tag{44}$$

Lemma 4. *If $m_s > 0$ we have*

$$\sum_{m=0}^{\infty} B^s_{c,\nu,m,\rho,\varepsilon,r} e^{2\pi i(m+m_s)(\tau-\frac{d}{c})} = \sum_{q=-\infty}^{\infty} e^{2\pi i m_s q} \left(-i(c\tau+d-cq)\right)^r \sum_{p=r+1}^{\infty} \frac{1}{p!} \left(\frac{2\pi i(\nu-m_s)}{c^2\tau+cd-cq} \right)^p \tag{45}$$

and if $m_s = 0$ we have

$$\sum_{m=0}^{\infty} B^{s}_{c,\nu,m,\rho,\varepsilon,r} e^{2\pi i(m+m_s)(\tau-\frac{d}{c})} = \frac{1}{2}\left(\frac{2\pi\nu}{c}\right)^{r+1}\frac{1}{(r+1)!}+ \lim_{N\to\infty}\sum_{q=-N}^{N}(-i(c\tau+d-cq))^{r}\sum_{p=r+1}^{\infty}\frac{1}{p!}\left(\frac{2\pi i\nu}{c^2\tau+cd-cq}\right)^{p} \tag{46}$$

Proof. It is a simply application of the Lipschitz formula and the power series expression for I_{r+1}, in the same way done by Knopp in [1]. □

The above result implies that

$$F_\nu(\tau) \tag{47}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} + K + \sum_{c=1}^{\infty}\sum_{\substack{d,\,c \\ 0\le d<c \\ (d,\,c)=1}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d}) \tag{48}$$

$$\begin{pmatrix} e^{2\pi i\frac{d'(m_1-\nu)}{c}}\lim_{N\to\infty}\sum_{q=-N}^{N} e^{2\pi i m_1 q}(-i(c\tau+d-cq))^{r}\sum_{p=r+1}^{\infty}\frac{1}{p!}\left(\frac{2\pi i(\nu-m_1)}{c^2\tau+cd-cq}\right)^{p} \\ \vdots \\ e^{2\pi i\frac{d'(m_p-\nu)}{c}}\lim_{N\to\infty}\sum_{q=-N}^{N} e^{2\pi i m_p q}(-i(c\tau+d-cq))^{r}\sum_{p=r+1}^{\infty}\frac{1}{p!}\left(\frac{2\pi i(\nu-m_p)}{c^2\tau+cd-cq}\right)^{p} \end{pmatrix} \tag{49}$$

where K is a column vector independent of τ given by

$$K = \frac{1}{2}\sum_{c=1}^{\infty}\sum_{\substack{d,\,c \\ 0\le d<c \\ (d,\,c)=1}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d}) \begin{pmatrix} e^{2\pi i\frac{d'(m_1-\nu)}{c}}\varphi(1) \\ \vdots \\ e^{2\pi i\frac{d'(m_p-\nu)}{c}}\varphi(p) \end{pmatrix} \tag{50}$$

where

$$\varphi(j) = \left(\frac{2\pi\nu}{c}\right)^{r+1}\frac{1}{(r+1)!} \quad \text{if } m_j = 0 \tag{51}$$

$$= 0 \quad \text{if } m_j > 0 \tag{52}$$

Now let $G_\nu(\tau) = F_\nu(\tau) - K$ then

$$G_\nu(\tau)(j) = e^{2\pi i(m_j-\nu)\tau} + \sum_{c=1}^{\infty}\sum_{\substack{d,\,c \\ 0\le d<c \\ (d,\,c)=1}} \varepsilon^{-1}(V_{c,d})\sum_{s=1}^{p} x_{js}(c,d)e^{2\pi i\frac{d'(m_s-\nu)}{c}} \tag{53}$$

$$\lim_{N\to\infty}\sum_{q=-N}^{N} e^{2\pi i m_s q}(-i(c\tau+d-cq))^{r}\sum_{p=r+1}^{\infty}\frac{1}{p!}\left(\frac{2\pi i(\nu-m_s)}{c^2\tau+cd-cq}\right)^{p} \tag{54}$$

now let $\delta = d - cq$, as q runs through all integers and as d runs through the set

$$D_c = \left\{ d | \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma(1), \text{ with } 0 \le -d < c \right\} \tag{55}$$

δ assumes exactly once, each value $\delta \in D^c = \{\delta \in \mathbb{Z} | (c, \delta) = 1\}$, and we can define $V_{c,\delta} = S^{-q} V_{c,d}$, we have

$$\varepsilon^{-1}(Vc, d)\rho^{-1}(V_{c,d}) \tag{56}$$

$$= \varepsilon^{-1}(Vc, d) \begin{pmatrix} x_{11}(c,d) & \cdots & x_{1p}(c,d) \\ \vdots & \ddots & \vdots \\ x_{p1}(c,d) & \cdots & x_{pp}(c,d) \end{pmatrix} \tag{57}$$

$$= \varepsilon^{-1}(V_{c,\delta} S^q)\rho^{-1}(V_{c,\delta} S^q) \tag{58}$$

$$= \varepsilon^{-1}(V_{c,\delta}) \begin{pmatrix} e^{-2\pi i m_1 q} & & \\ & \ddots & \\ & & e^{-2\pi i m_p q} \end{pmatrix} \rho^{-1}(V_{c,\delta}) \tag{59}$$

$$= \varepsilon^{-1}(V_{c,\delta}) \begin{pmatrix} e^{-2\pi i m_1 q} & & \\ & \ddots & \\ & & e^{-2\pi i m_p q} \end{pmatrix} \begin{pmatrix} x_{11}(c,\delta) & \cdots & x_{1p}(c,\delta) \\ \vdots & \ddots & \vdots \\ x_{p1}(c,\delta) & \cdots & x_{pp}(c,\delta) \end{pmatrix} \tag{60}$$

and therefore

$$\varepsilon^{-1}(V_{c,d}) x_{js}(c,d) = \varepsilon^{-1}(V_{c,\delta}) e^{-2\pi i m_s q} x_{js}(c,\delta) \tag{61}$$

Note also that $\delta' = d'$, and then we have

$$e^{-\frac{2\pi i \delta'(\nu - m_s)}{c}} = e^{-\frac{2\pi i d'(\nu - m_s)}{c}} \tag{62}$$

now we can rewrite $G_\nu(\tau)(j)$, but instead of writing δ we write d

$$\begin{aligned}
&G_\nu(\tau)(j) \\
&= e^{2\pi i(m_j-\nu)\tau} + \sum_{s=1}^{p}\sum_{c=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{d\,=\,-N\\ d\,\in\, D^c}}^{N} \varepsilon^{-1}(V_{c,d})x_{js}(c,d)e^{2\pi i\frac{d'(m_s-\nu)}{c}} \\
&\quad (-i(c\tau+d))^r \sum_{p=r+1}^{\infty}\frac{1}{p!}\left(\frac{2\pi i(\nu-m_s)}{c(c\tau+d)}\right)^p \\
&= e^{2\pi i(m_j-\nu)\tau} + \sum_{s=1}^{p}\sum_{c=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{d\,=\,-N\\ d\,\in\, D^c}}^{N} \varepsilon^{-1}(V_{c,d})x_{js}(c,d)e^{2\pi i\frac{d'(m_s-\nu)}{c}} \\
&\quad (-i(c\tau+d))^r \frac{1}{(r+1)!}\left(\frac{2\pi i(\nu-m_s)}{c(c\tau+d)}\right)^{r+1} \\
&\quad + \sum_{s=1}^{p}\sum_{c=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{d\,=\,-N\\ d\,\in\, D^c}}^{N} \varepsilon^{-1}(V_{c,d})x_{js}(c,d)e^{2\pi i\frac{d'(m_s-\nu)}{c}} \\
&\quad (-i(c\tau+d))^r \sum_{p=r+2}^{\infty}\frac{1}{p!}\left(\frac{2\pi i(\nu-m_s)}{c(c\tau+d)}\right)^p
\end{aligned}$$

The separation into two sums is justified since the first converges by Lemma (2.10) in [1], which is applicable since $\tau = iy$ with $y > 0$, and the second is absolutely convergent triple sum, which can be proven by using Lemma (2.5) in [1], and therefore the second sum can be rearranged in any manner.

Now let

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1), \quad \alpha > 0, \quad \beta < 0 \quad \delta > \gamma > 0 \tag{63}$$

and $t = (\gamma - \frac{\beta}{2})\delta^{-1}$, and define $\mathcal{J}_V(K)$ to be the trapezoid in the c-d plane bounded by the lines

$$c = 0, \quad \alpha c + \gamma d = tK, \quad \delta d + \beta c = \pm K \tag{64}$$

we obtain

$$
\begin{aligned}
&G_\nu(\tau)(j)\\
&= e^{2\pi i(m_j-\nu)\tau} + \sum_{s=1}^{p} \lim_{K\to\infty} \sum_{c=1}^{\infty} \sum_{\substack{(c,\,d)\,\in\,\mathcal{J}_V(K)\\ d\,\in\,D^c}} \varepsilon^{-1}(V_{c,d}) x_{js}(c,d) e^{2\pi i \frac{d'(m_s-\nu)}{c}}\\
&\qquad (-i(c\tau+d))^r \frac{1}{(r+1)!}\left(\frac{2\pi i(\nu-m_s)}{c(c\tau+d)}\right)^{r+1}\\
&\qquad + \sum_{s=1}^{p} \lim_{K\to\infty} \sum_{c=1}^{\infty} \sum_{\substack{(c,\,d)\,\in\,\mathcal{J}_V(K)\\ d\,\in\,D^c}} \varepsilon^{-1}(V_{c,d}) x_{js}(c,d) e^{2\pi i \frac{d'(m_s-\nu)}{c}}\\
&\qquad (-i(c\tau+d))^r \sum_{p=r+2}^{\infty} \frac{1}{p!}\left(\frac{2\pi i(\nu-m_s)}{c(c\tau+d)}\right)^{p}\\
&= e^{2\pi i(m_j-\nu)\tau} + \sum_{s=1}^{p} \lim_{K\to\infty} \sum_{c=1}^{\infty} \sum_{\substack{(c,\,d)\,\in\,\mathcal{J}_V(K)\\ d\,\in\,D^c}} \varepsilon^{-1}(V_{c,d}) x_{js}(c,d) e^{2\pi i \frac{d'(m_s-\nu)}{c}}\\
&\qquad (-i(c\tau+d))^r \left(e^{\frac{2\pi i(\nu-m_s)}{c(c\tau+d)}} - \sum_{p=0}^{r} \frac{1}{p!}\left(\frac{2\pi i(\nu-m_s)}{c(c\tau+d)}\right)^{p}\right)
\end{aligned}
$$

Now let

$$S_{\nu,K}(\tau) \tag{65}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} + \sum_{c=1}^{\infty} \sum_{\substack{(c,\,d)\,\in\,\mathcal{J}_V(K)\\ d\,\in\,D^c}} \varepsilon^{-1}(V_{c,d}) \rho^{-1}(V_{c,d}) \left(-i(c\tau+d)\right)^r \tag{66}$$

$$\begin{pmatrix} e^{2\pi i \frac{d'(m_1-\nu)}{c}} e^{\frac{2\pi i(\nu-m_1)}{c(c\tau+d)}} \\ \vdots \\ e^{2\pi i \frac{d'(m_p-\nu)}{c}} e^{\frac{2\pi i(\nu-m_p)}{c(c\tau+d)}} \end{pmatrix} \tag{67}$$

and since $V_{c,d} \in SL(2,\mathbb{Z})$ we can see that

$$e^{2\pi i \frac{d'(m_s-\nu)}{c}} e^{\frac{2\pi i(\nu-m_s)}{c(c\tau+d)}} = e^{2\pi i(m_s-\nu)V_{c,d}\tau} \tag{68}$$

and therefore

$$S_{\nu,K}(\tau) \tag{69}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} + \sum_{c=1}^{\infty} \sum_{\substack{(c,\,d)\,\in\,\mathcal{J}_V(K) \\ d\,\in\,D^c}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d})\left(-i(c\tau+d)\right)^r \tag{70}$$

$$\begin{pmatrix} e^{2\pi i(m_1-\nu)V_{c,d}\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)V_{c,d}\tau} \end{pmatrix} \tag{71}$$

now we can include the first term in the summation, since $\varepsilon(I)\rho(I)(-i)^{-r} = I$ and we have that

$$\begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} = \varepsilon^{-1}(I)\rho^{-1}(I)(-i)^r \begin{pmatrix} e^{2\pi i(m_1-\nu)I\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)I\tau} \end{pmatrix} \tag{72}$$

now we can include the pair $(c,d) = (0,1)$ and therefore we get

$$S_{\nu,K}(\tau) \;=\; \sum_{c=0}^{\infty} \sum_{\substack{(c,\,d)\,\in\,\mathcal{J}_V(K) \\ d\,\in\,D^c \\ (c,\,d)\,\neq\,(0,\,-1)}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d})\left(-i(c\tau+d)\right)^r \tag{73}$$

$$\begin{pmatrix} e^{2\pi i(m_1-\nu)V_{c,d}\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)V_{c,d}\tau} \end{pmatrix} \tag{74}$$

we can now extend the region $\mathcal{J}_V(K)$ to $\mathcal{P}_V(K)$, consisting in reflecting $\mathcal{J}_V(K)$ through the origin by including with $V_{c,d}$ and $-V_{c,d}$, and since we have that

$$\varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d})\left(-i(c\tau+d)\right)^r = \varepsilon^{-1}(-V_{c,d})\rho^{-1}(-V_{c,d})\left(-i(-c\tau-d)\right)^r \tag{75}$$

and

$$e^{2\pi i(m_j-\nu)V_{c,d}\tau} = e^{2\pi i(m_j-\nu)-V_{c,d}\tau} \tag{76}$$

we see that if we make the summation over $\mathcal{P}_V(K)$, every term of $S_{\nu,K}(\tau)$ occurs twice and therefore

$$S_{\nu,K}(\tau) \;=\; \frac{1}{2}\sum_{c\in\mathbb{Z}} \sum_{\substack{(c,\,d)\,\in\,\mathcal{P}_V(K) \\ d\,\in\,D^c}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d})\left(-i(c\tau+d)\right)^r \tag{77}$$

$$\begin{pmatrix} e^{2\pi i(m_1-\nu)V_{c,d}\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)V_{c,d}\tau} \end{pmatrix} \tag{78}$$

now $\mathcal{P}_V(K)$ is bounded by

$$\alpha c + \gamma d = \pm tK, \quad \delta d + \beta c = \pm K \tag{79}$$

Therefore we get

$$\varepsilon^{-1}(V)\rho^{-1}(V)\left(-i(\gamma\tau+\delta)\right)^r S_{\nu,K}(V\tau) \tag{80}$$

$$= \frac{1}{2}\sum_{c\in\mathbb{Z}} \sum_{\substack{(c,\,d)\,\in\,\mathcal{P}_V(K)\\ d\,\in\,D^c}} \varepsilon^{-1}(V)\varepsilon^{-1}(V_{c,d})\rho^{-1}(V)\rho^{-1}(V_{c,d}) \tag{81}$$

$$\left(-i(\gamma\tau+\delta)\right)^r\left(-i(cV\tau+d)\right)^r \begin{pmatrix} e^{2\pi i(m_1-\nu)V_{c,d}V\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)V_{c,d}V\tau} \end{pmatrix} \tag{82}$$

$$= \frac{1}{2}\sum_{c\in\mathbb{Z}} \sum_{\substack{(c,\,d)\,\in\,\mathcal{P}_V(K)\\ d\,\in\,D^c}} \varepsilon^{-1}(V_{c,d}V)\rho^{-1}(V_{c,d}V) \tag{83}$$

$$\left(-i((\alpha c+\gamma d)\tau+(\beta c+\delta d)\right)^r \begin{pmatrix} e^{2\pi i(m_1-\nu)V_{c,d}V\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)V_{c,d}V\tau} \end{pmatrix} \tag{84}$$

now we can make the transformation $c' = \alpha c + \gamma d$ and $d' = \beta c + \delta d$ which is a 1-1 transformation from $\mathcal{P}_V(K)$ to the rectangle

$$|c'| \leq tK, \qquad |d'| \leq K \tag{85}$$

also this map is a 1-1 correspondence between the pairs $\{(c,d)|c\in\mathbb{Z}, d\in D^c\}$

and the pairs $\{(c',d')|c'\in\mathbb{Z}, d'\in D^{c'}\}$, and then

$$\varepsilon^{-1}(V)\rho^{-1}(V)\left(-i(\gamma\tau+\delta)\right)^r S_{\nu,K}(V\tau) \tag{86}$$

$$= \frac{1}{2}\sum_{\substack{c'\in\mathbb{Z}\\|c'|\le tK}}\sum_{\substack{d'\in D^c\\|d'|\le K}} \varepsilon^{-1}(V_{c',d'})\rho^{-1}(V_{c',d'}) \tag{87}$$

$$(-i(c'\tau+d'))^r\begin{pmatrix} e^{2\pi i(m_1-\nu)V_{c',d'}\tau}\\ \vdots \\ e^{2\pi i(m_p-\nu)V_{c',d'}\tau}\end{pmatrix} \tag{88}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau}\\ \vdots \\ e^{2\pi i(m_p-\nu)\tau}\end{pmatrix} + \sum_{\substack{c\in\mathbb{Z}\\0<c\le tK}}\sum_{\substack{d\in D^c\\|d|\le K}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d}) \tag{89}$$

$$(-i(c\tau+d))^r\begin{pmatrix} e^{2\pi i(m_1-\nu)V_{c,d}\tau}\\ \vdots \\ e^{2\pi i(m_p-\nu)V_{c,d}\tau}\end{pmatrix} \tag{90}$$

now

$$\varepsilon^{-1}(V)\rho^{-1}(V)\left(-i(\gamma\tau+\delta)\right)^r G_{\nu}(V\tau) \tag{91}$$

$$= \lim_{k\to\infty}(\varepsilon^{-1}(V)\rho^{-1}(V)\left(-i(\gamma\tau+\delta)\right)^r S_{\nu,K}(V\tau) \tag{92}$$

$$-\sum_{c=1}^{\infty}\sum_{\substack{(c,\,d)\in\mathcal{J}_V(K)\\d\in D^c}} \varepsilon^{-1}(V)\varepsilon^{-1}(V_{c,d})\rho^{-1}(V)\rho^{-1}(V_{c,d})\left(-i(\gamma\tau+\delta)\right)^r \tag{93}$$

$$(-i(cV\tau+d))^r\begin{pmatrix} e^{2\pi i\frac{d'(m_1-\nu)}{c}}\sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i(\nu-m_1)}{c(cV\tau+d)}\right)^p\\ \vdots \\ e^{2\pi i\frac{d'(m_p-\nu)}{c}}\sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i(\nu-m_p)}{c(cV\tau+d)}\right)^p\end{pmatrix}) \tag{94}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau}\\ \vdots \\ e^{2\pi i(m_p-\nu)\tau}\end{pmatrix} + \lim_{k\to\infty}(\sum_{\substack{c\in\mathbb{Z}\\0<c\le tK}}\sum_{\substack{d\in D^c\\|d|\le K}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d})\left(-i(c\tau+d)\right)^r\begin{pmatrix} e^{2\pi i(m_1-\nu)V_{c,d}\tau}\\ \vdots \\ e^{2\pi i(m_p-\nu)V_{c,d}\tau}\end{pmatrix}) \tag{95}$$

$$-\sum_{c=1}^{\infty}\sum_{\substack{(c,\,d)\in\mathcal{J}_V(K)\\d\in D^c}} \varepsilon^{-1}(V_{c,d}V)\rho^{-1}(V_{c,d}V)\left(-i(\gamma\tau+\delta)\right)^r\left(-i(cV\tau+d)\right)^r \tag{96}$$

$$\begin{pmatrix} e^{2\pi i\frac{d'(m_1-\nu)}{c}}\sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i(\nu-m_1)}{c(cV\tau+d)}\right)^p\\ \vdots \\ e^{2\pi i\frac{d'(m_p-\nu)}{c}}\sum_{p=0}^{r}\frac{1}{p!}\left(\frac{2\pi i(\nu-m_p)}{c(cV\tau+d)}\right)^p\end{pmatrix}) \tag{97}$$

also using lemma (2.13) in [1].

$$G_\nu(\tau) \tag{98}$$

$$= \begin{pmatrix} e^{2\pi i(m_1-\nu)\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)\tau} \end{pmatrix} + \lim_{k\to\infty}\Big(\sum_{\substack{c\,\in\,\mathbb{Z} \\ 0\,<\,c\,\le\,tK}} \sum_{\substack{d\,\in\,D^c \\ |d|\,\le\,K}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d})\left(-i(c\tau+d)\right)^r \begin{pmatrix} e^{2\pi i(m_1-\nu)V_{c,d}\tau} \\ \vdots \\ e^{2\pi i(m_p-\nu)V_{c,d}\tau} \end{pmatrix} \tag{99}$$

$$- \sum_{\substack{c\,\in\,\mathbb{Z} \\ 0\,<\,c\,\le\,tK}} \sum_{\substack{d\,\in\,D^c \\ |d|\,\le\,K}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d})\left(-i(c\tau+d)\right)^r \begin{pmatrix} e^{2\pi i\frac{d'(m_1-\nu)}{c}} \sum_{p=0}^r \frac{1}{p!}\left(\frac{2\pi i(\nu-m_1)}{c(c\tau+d)}\right)^p \\ \vdots \\ e^{2\pi i\frac{d'(m_p-\nu)}{c}} \sum_{p=0}^r \frac{1}{p!}\left(\frac{2\pi i(\nu-m_p)}{c(c\tau+d)}\right)^p \end{pmatrix}\Big) \tag{100}$$

therefore

$$G_\nu(\tau) - \varepsilon^{-1}(V)\rho^{-1}(V)\left(-i(\gamma\tau+\delta)\right)^r G_\nu(V\tau) \tag{101}$$

$$= \lim_{k\to\infty}\Big(\sum_{c=1}^{\infty} \sum_{\substack{(c,\,d)\,\in\,\mathcal{J}_V(K) \\ d\,\in\,D^c}} \varepsilon^{-1}(V_{c,d}V)\rho^{-1}(V_{c,d}V)\left(-i(\gamma\tau+\delta)\right)^r \left(-i(cV\tau+d)\right)^r \tag{102}$$

$$\begin{pmatrix} e^{2\pi i\frac{d'(m_1-\nu)}{c}} \sum_{p=0}^r \frac{1}{p!}\left(\frac{2\pi i(\nu-m_1)}{c(cV\tau+d)}\right)^p \\ \vdots \\ e^{2\pi i\frac{d'(m_p-\nu)}{c}} \sum_{p=0}^r \frac{1}{p!}\left(\frac{2\pi i(\nu-m_p)}{c(cV\tau+d)}\right)^p \end{pmatrix} \tag{103}$$

$$- \sum_{\substack{c\,\in\,\mathbb{Z} \\ 0\,<\,c\,\le\,tK}} \sum_{\substack{d\,\in\,D^c \\ |d|\,\le\,K}} \varepsilon^{-1}(V_{c,d})\rho^{-1}(V_{c,d})\left(-i(c\tau+d)\right)^r \begin{pmatrix} e^{2\pi i\frac{d'(m_1-\nu)}{c}} \sum_{p=0}^r \frac{1}{p!}\left(\frac{2\pi i(\nu-m_1)}{c(c\tau+d)}\right)^p \\ \vdots \\ e^{2\pi i\frac{d'(m_p-\nu)}{c}} \sum_{p=0}^r \frac{1}{p!}\left(\frac{2\pi i(\nu-m_p)}{c(c\tau+d)}\right)^p \end{pmatrix} \tag{104}$$

now since the factor$(\gamma\tau+\delta)^r$ combine with $(cV\tau+d)^{r-p}$ produces a polynomial of degree at most r. On the other hand the limit of a sequence of polynomials of degree at most r converging at $r+1$ points is a polynomial of degree at most r.

Now since $F_\nu(\tau) = G_\nu(\tau) + K$, then we have that $F_\nu(V\tau) = G_\nu(V\tau) + K$, therefore we have that

$$F_\nu(\tau) - \varepsilon^{-1}(V)\rho^{-1}(V)\left(-i(\gamma\tau+\delta)\right)^r F_\nu(V\tau) = p_{V,\nu}(\tau, r, \varepsilon, \rho) \tag{105}$$

for

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1), \quad \alpha > 0, \quad \beta < 0 \quad \delta > \gamma > 0 \tag{106}$$

to prove it for every $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, first we see that if $M = S^q$, we have

$$F_\nu(S^q\tau) = \begin{pmatrix} e^{2\pi i q m_1} & & \\ & \ddots & \\ & & e^{2\pi i q m_p} \end{pmatrix} F_\nu(\tau) \tag{107}$$

which follows from the fourier expansion of $F_\nu(\tau)$, otherwise it is easy to show that exist $m, n \in \mathbb{Z}$ and $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$ such that

$$M = S^k V S^n, \quad \alpha > 0, \quad \beta < 0 \quad \delta > \gamma > 0 \quad \gamma = c, \quad \delta = d - cn \tag{108}$$

and therefore

$$F_\nu(M\tau) = F_\nu(S^k V S^n\tau) = \begin{pmatrix} e^{2\pi i k m_1} & & \\ & \ddots & \\ & & e^{2\pi i k m_p} \end{pmatrix} F_\nu(VS^n\tau) \tag{109}$$

$$= \begin{pmatrix} e^{2\pi i k m_1} & & \\ & \ddots & \\ & & e^{2\pi i k m_p} \end{pmatrix} \varepsilon(V)\rho(V)\left(-i(\gamma S^n\tau + \delta)\right)^r \tag{110}$$

$$(F_\nu(S^n\tau) - p_{V,\nu}(S^n\tau, \varepsilon, \rho)) \tag{111}$$

and since $\left(-i(\gamma S^n\tau + \delta)\right)^r = \left(-i(c\tau + d)\right)^r$, we have

$$F_\nu(\tau) - \begin{pmatrix} e^{-2\pi i n m_1} & & \\ & \ddots & \\ & & e^{-2\pi i n m_p} \end{pmatrix} p_{V,\nu}(S^n\tau, \varepsilon, \rho) \tag{112}$$

$$= \varepsilon^{-1}(V)\left(-i(c\tau + d)\right)^r \begin{pmatrix} e^{-2\pi i n m_1} & & \\ & \ddots & \\ & & e^{-2\pi i n m_p} \end{pmatrix} \tag{113}$$

$$\rho^{-1}(V) \begin{pmatrix} e^{-2\pi i k m_1} & & \\ & \ddots & \\ & & e^{-2\pi i k m_p} \end{pmatrix} F_\nu(M\tau) \tag{114}$$

$$= \varepsilon^{-1}(M)\rho^{-1}(M)\left(-i(c\tau + d)\right)^r F_\nu(M\tau) \tag{115}$$

and since

$$\begin{pmatrix} e^{-2\pi i n m_1} & & \\ & \ddots & \\ & & e^{-2\pi i n m_p} \end{pmatrix} p_{V,\nu}(S^n\tau, \varepsilon, \rho) \tag{116}$$

is a column vector of degree at most r the proof is complete □

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