Proposal

José Giménez

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The unrestricted partition function $p(n)$ counts the number of ways a positive integer n can be expressed as a sum of positive integers $\leq n$. The partition function is generated by Euler's infinite product

$$F(x) = \prod_{m=1}^{\infty} (1 - x^m)^{-1} = 1 + \sum_{n=1}^{\infty} p(n) x^n \qquad |x| < 1 \tag{1}$$

This establishes a relationship between the Dedekin Eta function defined as

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \qquad \tau \in H \tag{2}$$

and the partition function $p(n)$, given by

$$\eta^{-1}(\tau) = \sum_{m=-1}^{\infty} p(m+1) e^{2\pi i (m + \frac{23}{24})\tau} \qquad \tau \in H \tag{3}$$

Since $\eta(\tau)$ is a modular form of weight $1/2$, $\eta^{-1}(\tau)$ has weight $-1/2$, and the fact that $\eta^{-1}(\tau)$ is meromorphic and (3) makes $\eta^{-1}(\tau)$ a modular form of negative weight. From (1) we can see that

$$p(n) = \frac{1}{2\pi i} \int_C \frac{F(x)}{x^{n+1}} dx \tag{4}$$

where C is a positively oriented simple closed curve on the unit circle containing zero i its interior.

The circle method consist of making the change of variable $x = e^{2\pi i\tau}$ for every N and choose a path of integration $P(N)$ Joining i and $i+1$ through the upper arcs of the Ford circles of the Farey series F_N. An then from (4)

$$p(n) = \int_i^{i+1} F(e^{2\pi i\tau})e^{-2\pi in\tau}d\tau = \int_{P(N)} F(e^{2\pi i\tau})e^{-2\pi in\tau}d\tau \tag{5}$$

Rademacher [9] modified the circle method first introduced by Hardy and Ramanujan to construct a new path of integration C in order to get an exact formula for the partition function $p(n)$

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right\}}{\sqrt{n-\frac{1}{24}}}\right) \tag{6}$$

where

$$A_k(n) = \sum_{\substack{0\le h<k \\ (h,k)=1}} e^{\pi is(h,k)-2\pi inh/k} \tag{7}$$

Knopp and Mason [3] obtained growth conditions of the Fourier coefficients of vector-valued modular forms. In [4] they developed a general theory of vector-valued modular forms. The following definition is given in [4]: Let $F(\tau) = (F_1(\tau), \ldots, F_p(\tau))$ be a p-tuple of functions holomorphic in the complex upper half-plane H and $\rho : \Gamma \longrightarrow GL(p, C)$ a p-dimensional complex representation.(F, ρ), or simply F, is a vector-valued form of real weight k on the modular group $\Gamma = SL(2, Z)$ if

1. For all $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$(F_1(\tau), \ldots, F_p(\tau))^t \mid_k V(\tau) = \rho(V)(F_1(\tau), \ldots, F_p(\tau))^t \tag{8}$$

2. Each component function $F_j(\tau)$ has a convergent q-expansion meromorphic at infinity:

$$F_j(\tau) = \sum_{n \geq h_j} a_n(j) q^{\frac{n}{N_j}} \tag{9}$$

with N_j a positive integer and $q = e^{2\pi i \tau}$. The Slash operator $|_k\, V$ in (8) is defined by:

$$F \mid_k V(\tau) = F \mid_k^v V(\tau) = v(V)^{-1}(c\tau + d)^{-k} F(V\tau). \tag{10}$$

The goal of my thesis is to:

- Apply the circle method to vector-valued modular forms of negative weight in order to get the exact formula for the Fourier coefficients.
- Show that exists a vector-valued modular form of negative weight.
- Find an upper and lower bound in the dimension of $M(k, \rho, p)$, the space of vector-valued modular forms when k is negative.

References

[1] M. Knopp, Modular Functions in Analytic Number Theory, Markham Publishing, Illinois, 1970.

[2] M. Knopp and G. Mason, Generalized modular forms, J. Number Theory 99 (2003), 1-28.

[3] M. Knopp and G. Mason, On vector-valued modular forms and their Fourier coefficients, Acta Arith. 110 (2003), no. 2, 117124.

[4] M. Knopp and G. Mason, Vector-Valued Modular forms and Poincare Series, Illinois Journal of Mathematics 48 (2004), 1345-1366.

[5] H. Rademacher, On the partition function $p(n)$. Proc. London Math. Soc. 43 (1937), 241-254.

[6] H. Rademacher, A convergent series for the partition function $p(n)$. Proc. Nat. Ac. Sci USA 23 (1937), 78-84.

[7] H. Rademacher, The Fourier coefficients of the modular invariant $J(\tau)$. Am. J. Math. 60 (1938), 501-512.

[8] H. Rademacher, The Fourier series and the functional equation of the absolute modular invariant $J(\tau)$. Am. J. Math. 61 (1939), 237-248.

[9] H. Rademacher, On the expansion of the partition function in a series. Ann. Math. 44 (1943), 416-422.