Proposal

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The number $p(n)$ of unrestricted partitions of n appears in the generating function

$$F(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{m=1}^{\infty} (1 - x^m)^{-1}$$

$$\eta^{-1}(\tau) = \sum_{m=-1}^{\infty} p(m+1)e^{2\pi i(m+\frac{23}{24})\tau}$$

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

Rademacher [9] modified the circle method first introduced by Hardy and Ramanujan to construct a new path of integration C in order to get an exact formula for the partition function $p(n)$

$$p(n) = \frac{1}{2\pi i} \int_C \frac{F(x)}{x^{n+1}} dx$$

Knopp and Mason [3] obtained growth conditions of the Fourier coefficients of vector-valued modular forms. In [4] they developed a general theory of vector-valued modular forms. The following definition is given in [4]: Let $F(\tau) = (F_1(\tau), \ldots, F_p(\tau))$ be a p-tuple of functions holomorphic in the complex upper half-plane H and $\rho : \Gamma \longrightarrow GL(p, C)$ a p-dimensional complex representation.(F,ρ), or simply F, is a vector-valued form of real weight k on the modular group $\Gamma = SL(2,Z)$ if

1. For all $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$(F_1(\tau),\ldots,F_p(\tau))^t \mid_k V(\tau) = \rho(V)(F_1(\tau),\ldots,F_p(\tau))^t \tag{1}$$

2. Each component function $F_j(\tau)$ has a convergent q-expansion meromorphic at infinity:

$$F_j(\tau) = \sum_{n \geq h_j} a_n(j) q^{\frac{n}{N_j}} \tag{2}$$

with N_j a positive integer and $q = e^{2\pi i\tau}$. The Slash operator $|_k\ V$ in (1) is defined by:

$$F \mid_k V(\tau) = F \mid_k^v V(\tau) = v(V)^{-1}(c\tau + d)^{-k} F(V\tau). \tag{3}$$

The goal of my thesis is to:

- Apply the circle method to vector-valued modular forms of negative weight in order to get the exact formula for the Fourier coefficients.
- Show that exists a vector-valued modular form of negative weight.
- Find an upper and lower bound in the dimension of $M(k,\rho,p)$, the space of vector-valued modular forms when k is negative.

References

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