Fourier coefficients of vector-valued modular forms of negative weight and Eichler cohomology

A Dissertation
Submitted to
the Temple University Graduate Board

in Partial Fulfillment
of the Requirements for the Degree of
DOCTOR OF PHILOSOPHY

by
José Giménez
May, 2007

ACKNOWLEDGEMENTS

I want to express my gratitude to all people that helped me in one way or another to finish my thesis successfully.

I want to express my deep appreciation to my thesis advisor, Professor Marvin Knopp, for putting a huge amount of time, energy and dedication into this research: thank you!. I am honored to be a student of such a great mathematician.

To Professor Boris Datskovsky, for serving on my examining committee and for being one of the best teachers I have had, with deep knowledge in almost every branch of mathematics.

To Professors Sinai Robins and Wladimir Pribitkin, for serving on my examining committee, for being very helpful in the whole process of the defense.

To my best friend and mathematical sibling Wissam Raji, for proving that he is a great friend at every opportunity he had in the last 4 years.

To my other mathematical siblings Karen Taylor and Henok Mawi.

To my friends and colleagues Marilena Downing and Gabriel Soloff, for their support and inspiration.

To my high school teacher Pep Solé. I used to be the worst student of mathematics in the class until ninth grade. His brilliant teaching was so magnetic that I became all of a sudden the best student of mathematics in tenth grade to the astonishment of my previous teachers. He is the one who made me love mathematics.

Last but not least to my eleven siblings. In order of appearance: Juan, Jesús, Mercedes, Teresa, Begoña, Mariángeles, Pilar, Santiago, Elisa, Javier and Rafael.

To my parents, Graciano and Mariángeles.

With love and admiration

TABLE OF CONTENTS

CHAPTER 1

Fourier Coefficients of Vector-Valued Modular Forms of negative weight

1.1 Outline of the thesis

In Chapter 1 we will use the circle method as devised by Rademacher and Zuckerman in [19] to obtain a formula (1.106) for the coefficients of the Fourier expansion of a vector-valued modular form of negative weight that depends on the principal part only. In Section 1.3 we get an estimate for the length of a linear transformation in $\Gamma(1)$, that depends on the Euclidean algorithm. In Chapter 2 we will see that if we apply the formula (1.106) with a given set of column vectors, a representation ρ on $\Gamma(1)$ and a multiplier system v on $\Gamma(1)$ in weight $-k$ for $k \in \mathbb{Z}^+$ big enough, we do not necessarily get a vector-valued modular form, but we get a vector-valued Eichler integral. This generalizes the work of Knopp in [6]. In Section 2.2, generalizing the work of Knopp in [6] p. 183, we describe a way to construct vector-valued modular forms of negative weight. In Section 2.3 we will define the vector-valued supplementary series in the same way as Knopp did for the scalar case in [6] to get the generalization of Theorem 4.9 of [6] to the vector-valued case. In Chapter 3 we generalize the Eichler cohomology to the vector-valued case. In Section 3.2 we generalize Theorem 3 of Husseini-Knopp [4] to obtain an Eichler cohomology theorem for the vector-valued case. In Section 3.3 we state a generalization of Theorem 1 of [4]. We also state the same theorem restricted to the parabolic cohomology. We conclude by using the generalized Poincare series [13, p. 164] to prove the theorem when restricted to the parabolic cohomology.

1.2 Introduction

Knopp and Mason [11] obtained growth conditions for the Fourier coefficients of vector-valued modular forms of positive weight. In [12] they developed a general theory of vector-valued modular forms.

Let $F(\tau) = (F^{(1)}(\tau), \ldots, F^{(p)}(\tau))^t$ be a p-tuple of functions holomorphic in the complex upper half-plane $\mathcal{H}$ and $\rho : \Gamma \longrightarrow GL(p, \mathbb{C})$ a p-dimensional complex representation (F, ρ), or simply F, is a vector-valued form of real weight $-k$ on the modular group $\Gamma = SL(2, \mathbb{Z})$ if

1. for all $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$(F^{(1)}(\tau), \ldots, F^{(p)}(\tau))^t \mid_{-k,v,\rho} V(\tau) = (F^{(1)}(\tau), \ldots, F^{(p)}(\tau))^t; \qquad (1.1)$$

2. each component function $F^{(j)}(\tau)$ has a convergent q-expansion meromorphic at infinity:

$$F^{(j)}(\tau) = q^{m_j} \sum_{\nu \geq \mu_j} a_\nu^{(j)} q^\nu \qquad (1.2)$$

   with $0 \leq m_j < 1$ a positive rational number, μ_j an integer and $q = e^{2\pi i \tau}$.

The slash operator $\mid_{-k,v,\rho} V$ in (1.1) is defined by:

$$F \mid_{-k,v,\rho} V(\tau) = v(V)^{-1} (c\tau + d)^k \rho^{-1}(V) F(V\tau), \qquad (1.3)$$

where v is a classical multiplier system on Γ of weight $-k$. Thus $v(V)$ is a complex number independent of τ such that

1. $|v(V)| = 1$ for all $V \in \Gamma$,

2. v satisfies the consistency condition

$$v(V_3)(c_3\tau + d_3)^{-k} = v(V_1)v(V_2)(c_1V_2\tau + d_1)^{-k}(c_2\tau + d_2)^{-k}, \qquad (1.4)$$

where $V_3 = V_1V_2$ and $V_i = \begin{pmatrix} * & * \\ c_i & d_i \end{pmatrix}$, $i = 1, 2$ and 3, and

3. v satisfies the nontriviality condition

$$v(-I) = e^{\pi ik}. \qquad (1.5)$$

In [11] Knopp and Mason show that the representation ρ can be normalized so that

$$v\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} e^{2\pi im_1} & & \\ & \ddots & \\ & & e^{2\pi im_p} \end{pmatrix}, \qquad (1.6)$$

where $0 \leq m_j < 1$, $m_j \in \mathbb{Q}$ for $1 \leq j \leq p$. These are the m_j's given in the Fourier expansion (1.2). In [11] Knopp and Mason assume $0 < m_j \leq 1$. We prefer to change the interval for convenience. In the rest of the paper we will assume that $-\pi \leq \arg\omega < \pi$ for $\omega \in \mathbb{C}$, $\omega \neq 0$.

In the remainder of Chapter 1, we will generalize the method of [19] to find the Fourier coefficients of vector-valued modular forms of sufficiently negative weight (Theorem 1.9). We need to introduce some concepts:

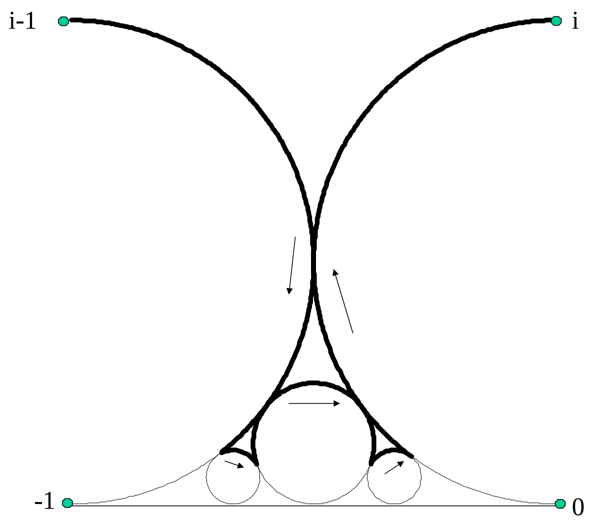
Definition 1.1 *The set of Farey fractions of order N, denoted by F_N, is the set of irreducible fractions in the interval $[-1, 0]$ with denominator smaller than or equal to N.*

Definition 1.2 *Given a rational number $\frac{d}{c}$, where $(c, d) = 1$, the Ford circle $C(c, d)$ associated to this fraction is a circle in the complex plane with radius $\frac{1}{2c^2}$ and center at the point $\frac{d}{c} + \frac{i}{2c^2}$.*

An important fact about Ford circles is that two consecutive Farey fractions in F_N give rise to tangent Ford circles. Moreover two Ford circles intersect only if the corresponding Farey fractions are consecutive in F_N, for some N.

Definition 1.3 *Let $\frac{d_1}{c_1} < \frac{d}{c} < \frac{d_2}{c_2}$ be three consecutive Farey fractions in F_N. We define $\chi_{c,d}(N)$ as the arc of $C(c,d)$ that joins the point of tangency of $C(c_1,d_1)$ with $C(c,d)$ and the point of tangency of $C(c,d)$ with $C(c_2,d_2)$, as we move clockwise in $C(c,d)$.*

Definition 1.4 *The Rademacher path of integration $P(N)$ is the path joining $i-1$ with i by moving clockwise from the arc $\chi_{c,d}(N)$ to the arc $\chi_{c',d'}(N)$, where $\frac{c}{d}$ and $\frac{c'}{d'}$ are two consecutive Farey fractions in F_N.*



Definition 1.5 *The Farey dissection of the circle $C(N)$, given by $|x| = e^{-2\pi N^{-2}}$, denoted by $F_{diss}(N)$, is a path around zero given by*

$$e^{-2\pi N^{-2}} e^{2\pi i P(N)}. \tag{1.7}$$

Now, by the Cauchy formula, if $f(x)$ is analytic in the unit circle except possibly at zero where there could be a singularity, we get

$$\begin{aligned} a_m &= \frac{1}{2\pi i} \int_{C(N)} \frac{f(x)}{x^{m+1}} dx \\ &= \frac{1}{2\pi i} \int_{F_{diss}(N)} \frac{f(x)}{x^{m+1}} dx \\ &= \frac{1}{2\pi i} \sum_{\frac{d}{c} \in F_N} \int_{\xi_{c,d}(N)} \frac{f(x)}{x^{m+1}} dx, \end{aligned} \tag{1.8}$$

where $\xi_{c,d}(N)$ is the path given by

$$e^{-2\pi N^{-2}} e^{2\pi i \chi_{c,d}(N)}. \tag{1.9}$$

In the remainder of Chapter 1, we will be interested in constructing elements of Γ from Farey fractions $\frac{d}{c}$ in F_N, as follows. Since $(c, d) = 1$ we can easily find an unique integer a, such that

$$ad \equiv 1 \pmod{c} \tag{1.10}$$

and

$$0 \leq a < c. \tag{1.11}$$

We also define $b = \frac{ad-1}{c}$ which is an integer since (1.10) applies. Therefore, given a Farey fraction $\frac{d}{c}$ in F_N there exists a unique $V \in \Gamma$ of the form

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad c > a \geq 0, c > -d \geq 0. \tag{1.12}$$

Notice that under the above conditions, since $V \in \Gamma$, we have the following:

1. $b \leq 0$. For,

$$ad - bc = 1 \Rightarrow bc = ad - 1, \tag{1.13}$$

   and since $c > 0$, $d \leq 0$ and $a \geq 0$, we have that $b \leq 0$.

2. If $d \neq 0$ then $c > -d \geq -b \geq 0$. Assume that $-b > -d$. Then

$$ad - bc > d(a - c) \geq 0. \tag{1.14}$$

The last inequality holds since $b < d < 0$ and $c > a$. $d(a-c)$ will be zero only if $d = 0$. But we assumed that $d \neq 0$. Therefore $d(a-c) \geq 1$, and $ad - bc > 1$, which is a contradiction.

3. If $d = 0$, then $b = -1$ and $c = 1$.

Therefore,

$$b = \frac{ad-1}{c}, \quad c > -d \geq -b \geq 0. \tag{1.15}$$

Also given a vector-valued modular form (F, ρ), for $x = e^{2\pi i\tau}$ we define

$$f^{(j)}(x) = x^{-m_j} F^{(j)}(\tau) = \sum_{\nu \geq \mu_j}^{\infty} a_\nu^{(j)} x^\nu, \quad 1 \leq j \leq p, \quad a_{\mu_j}^{(j)} \neq 0, \tag{1.16}$$

which is analytic in the unit circle and has a zero at $x = 0$ of order μ_j, if $\mu_j > 0$, or a pole of order $-\mu_j$, if $\mu_j < 0$. We will use this function to get the coefficients $a_\nu^{(j)}$ of the vector-valued modular form by way of the Cauchy's formula.

1.3 A new estimate for $L(V)$

Henceforth we will assume without loss of generality that $c \geq 0$ for $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma(1)$, since V and $-V$ give the same transformation.

Lemma 1.1 *Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma(1)$, such that $c \geq d \geq 0$ and let s be the number of steps in the Euclidean algorithm applied to the pair c, d. Then*

1. *if s is odd, we can write*

$$V = T^m \begin{pmatrix} 1 & 0 \\ q_s & 1 \end{pmatrix} \begin{pmatrix} 1 & q_{s-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_{s-2} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix}, \tag{1.17}$$

and

2. if s is even, we can write

$$V = T^m S \begin{pmatrix} 1 & q_s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_{s-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & q_{s-2} \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix}, \tag{1.18}$$

where $q_1, \ldots, q_s \in \mathbb{Z}^+$ *are the successive quotients given by applying the Euclidean Algorithm to the pair* c, d*.*

Proof 1.1

Therefore since $V \in \Gamma(1)$ we have that there exists an integer m such that

$$V = \begin{pmatrix} m+1 & m \\ 1 & 1 \end{pmatrix} = T^m \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1.19}$$

1. Let $c > d \geq 0$ and s odd. We will prove (1.17) by induction. Since $V \in \Gamma(1)$, we already know that $(c, d) = 1$. Now, if $s = 1$, then

$$c = r_0, \quad d = r_1 \quad \text{and} \quad c = q_1 d, \tag{1.20}$$

and since $(c, d) = 1$, we have that $d = 1$. It is then easy to check that

$$V = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix} = T^b \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} * & b \\ c & 1 \end{pmatrix}, \tag{1.21}$$

and therefore (1.17) follows for $s = 1$ if we put $m = b$ and $q_1 = c$.

Now assume that (1.17) holds for all transformations in which $c > d \geq 0$ and for $s \leq s^*$ odd. We want to show that (1.17) will also hold if the number of steps in applying the Euclidean algorithm to the pair c, d is $s^* + 2$, provided that $c > d \geq 0$. In this case we have that

$$c = r_0, \quad d = r_1, \quad c = q_1 d + r_2 \quad \text{and} \quad d = q_2 r_2 + r_3, \quad \text{for} \quad r_2 > r_3 \geq 0, \tag{1.22}$$

and the number of steps required to apply the Euclidean algorithm to the pair r_2, r_3 is s^*. It is easy to check that

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ r_2 & r_3 \end{pmatrix} \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix}. \tag{1.23}$$

Thus, by the inductive hypothesis, and since $r_2 > r_3 \geq 0$, it follows that we can write the matrix $\begin{pmatrix} * & * \\ r_2 & r_3 \end{pmatrix}$ in terms of the q_j's as in (1.17). This combined with (1.23) proves the lemma for s odd.

2. Let $c > d \geq 0$ and s even. We prove (1.18) first for $s = 2$. If $s = 2$, then we have that $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where

$$c = r_0, \quad d = r_1, \quad c = q_1 d + r_2 \quad \text{and} \quad d = q_2 r_2, \tag{1.24}$$

and since $(c, d) = 1$, we have that $r_2 = 1$ and $d = q_2$. It is clear that

$$S \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} = S \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} = \begin{pmatrix} -q_1 & -1 \\ c & d \end{pmatrix}. \tag{1.25}$$

Since the lower row of V and the lower row of (1.25)are the same, and since both are in $SL(2, \mathbb{Z})$, we have that $a \equiv -q_1 \pmod{c}$. Therefore there exists an m such that $a = -q_1 + cm$, and

$$T^m S \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} = \begin{pmatrix} -q_1 + cm & * \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = V, \tag{1.26}$$

as claimed.

Now assume that (1.18) holds for for all transformations in which $c > d \geq 0$ and for $s \leq s^*$ even. We want to show that (1.18) will also hold if the number of steps in applying the Euclidean algorithm to the pair c, d is $s^* + 2$, provided that $c > d \geq 0$. In this case we have that

$$c = r_0, \quad d = r_1, \quad c = q_1 d + r_2 \quad \text{and} \quad d = q_2 r_2 + r_3, \quad \text{for} \quad r_2 > r_3 \geq 0, \tag{1.27}$$

and the number of steps required to apply the Euclidean algorithm to the pair r_2, r_3 is s^*. It is easy to check that

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ r_2 & r_3 \end{pmatrix} \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix}. \tag{1.28}$$

Thus, by the inductive hypothesis, and since $r_2 > r_3 \geq 0$, we conclude that we can write the matrix $\begin{pmatrix} * & * \\ r_2 & r_3 \end{pmatrix}$ in terms of the q_j's as in (1.18). This, combined with (1.28), proves the lemma for s even, and therefore the proof is complete.

We now consider the length of V with respect to the generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of Γ. Namely, we write V as a product $V = \pm V_1 \ldots V_L$, where each V_j is equal to either S or T^{n_j}, for some integer n_j, and no two consecutive V_j are both equal to S or a power of T. The way to write $V \in \Gamma$ is not unique since $(ST)^3 = -I$ and we can include the string $(ST)^3$ as many times as we want. However there exists a minimal length. Let $L(V)$ be the minimal length for V.

Lemma 1.2 *An upper bound for the minimal length of a transformation* $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *in* $\Gamma(1)$ *with* $c \geq 0$ *is*

$$L(V) \leq \frac{2}{\log \alpha}(\log c + 1) + 3, \tag{1.29}$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}. \tag{1.30}$$

Proof 1.2

Suppose first that $c \geq d \geq 0$. Then, since

$$\begin{pmatrix} 1 & 0 \\ q_j & 1 \end{pmatrix} = -ST^{-q_j}S, \tag{1.31}$$

by Lemma 1.1 we can write V as

$$V = \pm T^m ST^{(-1)^s q_s} \ldots [-ST^{-q_3}S]T^{q_2}[-ST^{-q_1}S]. \tag{1.32}$$

Therefore,

$$V = \pm T^m ST^{(-1)^s q_s} ST^{(-1)^{s-1} q_{s-1}} S \ldots T^{q_2} ST^{-q_1} S. \tag{1.33}$$

Therefore if $c \geq d \geq 0$, we have that

$$L(V) \leq 2s + 2. \tag{1.34}$$

If $d > c$ or $d < 0$, then there exists $n \in \mathbb{Z}$, such that

$$d = d' + nc, \quad 0 \leq d' < c. \tag{1.35}$$

Therefore

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ c & d' \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}. \tag{1.36}$$

Note that by (1.33) and (1.36), for all $V \in \Gamma(1)$, we have that

$$V = \pm T^m ST^{(-1)^s q_s} ST^{(-1)^{s-1} q_{s-1}} S \ldots T^{q_2} ST^{-q_1} ST^n. \tag{1.37}$$

Therefore by (1.34) and (1.36), we have that

$$L(V) \leq L\left(\begin{pmatrix} * & * \\ c & d' \end{pmatrix}\right) + 1 \leq 2s + 3. \tag{1.38}$$

Here s is the number of steps required to apply the Euclidean algorithm to the pair c and d'.

Now by Lame's theorem, as explained by Dixon in the introduction of [2], we have that,

$$s \leq \frac{\log c + 1}{\log \alpha}, \tag{1.39}$$

and by (1.34) and (1.38) we have that

$$L(V) \leq 2\frac{\log c + 1}{\log \alpha} + 3. \tag{1.40}$$

The proof is complete.

Now we are interested in bounding $\rho(V)$, where ρ is a representation on $\Gamma(1)$. Knopp and Mason [11] showed that

$$\mid \rho^{(j,m)}(V) \mid \leq p^{L(V)-1} K_1^{L(V)}, \quad 1 \leq m, j \leq p, \tag{1.41}$$

where $\rho(V) = (\rho^{(j,m)}(V))$ and K_1 is a constant that satisfies $\mid \rho^{(j,m)}(S) \mid \leq K_1$, for all $1 \leq m, j \leq p$. Therefore by Lemma 1.2 we have that

$$\mid \rho^{(j,m)}(V) \mid \leq K_2 c^{\delta}, \quad \delta = \frac{2}{\log \alpha} \log pK_1, \tag{1.42}$$

where K_2 is independent of V. Note that since the minimal length of V^{-1} is the same as the minimal length of V, we can use the same bound for

$$\mid \rho(V)^{-1} \mid = \mid \rho(V^{-1}) \mid . \tag{1.43}$$

In the remainder of this section we will prove some lemmas in order to show that if $q_1, \ldots, q_s \in \mathbb{Z}^+$, $n \in \mathbb{Z}$, such that

$$V = ST^{(-1)^s q_s} ST^{(-1)^{s-1} q_{s-1}} S \ldots T^{q_2} ST^{-q_1} ST^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.44}$$

and if for $0 \leq j \leq s$ we define the matrix

$$M_j = T^{(-1)^j q_j} ST^{(-1)^{j-1} q_{j-1}} S \ldots T^{q_2} ST^{-q_1} ST^n = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}, \tag{1.45}$$

with $M_0 = T^n$ and $SM_s = V$, then

$$|c| \geq |\alpha_j|, \quad |c| \geq |\gamma_j|, \quad |d| \geq |\beta_j| \quad \text{and} \quad |d| \geq |\delta_j|. \tag{1.46}$$

These inequalities will be very useful in Section 3.4 to show convergence of the generalized Poincare series (3.50).

Lemma 1.3 *Let $q_1, \ldots, q_s \in \mathbb{Z}^+$ and*

$$V = ST^{(-1)^s q_s} ST^{(-1)^{s-1} q_{s-1}} S \ldots T^{q_2} ST^{-q_1} S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.47}$$

Then if s is even we have

$$sgn(c) = sgn(d) = sgn(\text{-}a) = sgn(\text{-}b), \tag{1.48}$$

while for odd s, we have

$$sgn(c) = sgn(d) = sgn(a) = sgn(b). \tag{1.49}$$

Proof 1.3

Since

$$-ST^{-q_j}S = \begin{pmatrix} 1 & 0 \\ q_j & 1 \end{pmatrix}, \tag{1.50}$$

then

$$V = \pm ST^{(-1)^s q_s} \dots [-ST^{-q_3}S]T^{q_2}[-ST^{-q_1}S], \tag{1.51}$$

and we can rewrite V as

$$V = \begin{cases} \pm[-ST^{-q_s}S]T^{q_{s-1}} \dots [-ST^{-q_3}S]T^{q_2}[-ST^{-q_1}S], & s \text{ odd}, \\ \pm ST^{-q_s}[-ST^{-q_{s-1}}] \dots [-ST^{-q_3}S]T^{q_2}[-ST^{-q_1}S], & s \text{ even}, \end{cases} \tag{1.52}$$

or, what is the same,

$$V = \begin{cases} \pm \begin{pmatrix} 1 & 0 \\ q_s & 1 \end{pmatrix} \begin{pmatrix} 1 & q_{s-1} \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix}, & s \text{ odd} \\ \pm S \begin{pmatrix} 1 & q_s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_{s-1} & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix}, & s \text{ even}. \end{cases} \tag{1.53}$$

Now if s is odd, since all entries in

$$\begin{pmatrix} 1 & 0 \\ q_s & 1 \end{pmatrix} \begin{pmatrix} 1 & q_{s-1} \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} \tag{1.54}$$

are positive, (1.49) follows automatically. On the other hand since all entries in

$$\begin{pmatrix} 1 & q_s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_{s-1} & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} \tag{1.55}$$

are positive, then the upper and lower row of

$$S \begin{pmatrix} 1 & q_s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_{s-1} & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & q_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} \tag{1.56}$$

have different signs and (1.48) follows automatically. The proof is complete.

Lemma 1.4 *Let V as in Lemma 1.3, then $|c| \geq |d|$ and $|a| \geq |b|$.*

Proof 1.4

It is easily proven by induction on s.

Lemma 1.5 *Let* $q_1, \ldots, q_s \in \mathbb{Z}^+$ *and*

$$M = T^{(-1)^{s-1}q_{s-1}} S T^{(-1)^{s-2}q_{s-2}} S \ldots T^{q_2} S T^{-q_1} S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.57}$$

and

$$V = S T^{(-1)^s q_s} S M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.58}$$

Then

$$1. \quad |c| \geq |\gamma|, \quad |d| \geq |\delta|, \quad |c| \geq |\alpha| \quad \textit{and} \quad |d| \geq |\beta|, \tag{1.59}$$

$$2. \quad |c| \geq |a| \quad \textit{and} \quad |d| \geq |b|, \tag{1.60}$$

$$3. \quad |c-d| \geq |\gamma - \delta| \quad \textit{and} \quad |c-d| \geq |\alpha - \beta|. \tag{1.61}$$

Proof 1.5

First note that

$$|c| = |-\gamma + (-1)^s q_s \alpha| \quad \text{and} \quad |d| = |-\delta + (-1)^s q_s \beta|\,. \tag{1.62}$$

Now since $SM = \begin{pmatrix} -\gamma & -\delta \\ \alpha & \beta \end{pmatrix}$ has the form of the matrix in Lemma 1.3 with $q_1, \ldots, q_{s-1}$, we have that

$$\operatorname{sgn}(-\gamma) = \operatorname{sgn}((-1)^s \alpha) = \operatorname{sgn}(-\delta) = \operatorname{sgn}((-1)^s \beta). \tag{1.63}$$

Therefore we can rewrite (1.62) as

$$\begin{aligned} |c| &= |-\gamma| + |(-1)^s q_s \alpha| \\ &= |\gamma| + q_s\,|\alpha| \end{aligned} \tag{1.64}$$

and

$$\begin{aligned} |d| &= |-\delta| + |(-1)^s q_s \beta| \\ &= |\delta| + q_s\,|\beta|\,, \end{aligned} \tag{1.65}$$

and (1.59) follows.

Now since $a = -\alpha$ and $b = -\beta$, by (1.59) it is clear that

$$|c| \geq |\alpha| = |a|, \quad \text{and} \quad |d| \geq |\beta| = |b| \tag{1.66}$$

and (1.60) follows.

Now by (1.63) and Lemma 1.4 applied to $SM = \begin{pmatrix} -\gamma & -\delta \\ \alpha & \beta \end{pmatrix}$ with $q_1, \ldots, q_{s-1}$, we have that

$$sgn(-\gamma) = sgn(-\gamma + \delta) \quad \text{and} \quad sgn((-1)^s \alpha) = sgn\left((-1)^s (\alpha - \beta)\right). \tag{1.67}$$

Therefore by (1.67) and (1.63) we have that

$$sgn(-\gamma + \delta) = sgn\left((-1)^s (\alpha - \beta)\right). \tag{1.68}$$

Thus,

$$\begin{aligned} |c - d| &= |-\gamma + (-1)^s q_s \alpha - (-\delta + (-1)^s q_s \beta)| \\ &= |-\gamma + \delta| + |(-1)^s q_s (\alpha - \beta)| \\ &= |\gamma - \delta| + q_s |\alpha - \beta| \end{aligned} \tag{1.69}$$

and (1.61) follows. The proof is complete.

Lemma 1.6 *Let* $q_1, \ldots, q_s \in \mathbb{Z}^+$, $n \in \mathbb{Z}$,

$$M = T^{(-1)^{s-1} q_{s-1}} S T^{(-1)^{s-2} q_{s-2}} S \ldots T^{q_2} S T^{-q_1} S T^n = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.70}$$

and

$$V = S T^{(-1)^s q_s} S M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.71}$$

Then

$$1. \quad |c| \geq |\gamma| \quad \textit{and} \quad |c| \geq |\alpha|, \tag{1.72}$$

$$2. \quad |d| \geq |\delta| \quad \textit{and} \quad |d| \geq |\beta|. \tag{1.73}$$

Proof 1.6

Put

$$M' = T^{(-1)^{s-1}q_{s-1}}ST^{(-1)^{s-2}q_{s-2}}S\ldots T^{q_2}ST^{-q_1}S = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \tag{1.74}$$

and

$$V' = ST^{(-1)^s q_s}SM' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \tag{1.75}$$

so that

$$M = M'T^n = \begin{pmatrix} \alpha' & \alpha'n+\beta' \\ \gamma' & \gamma'n+\delta' \end{pmatrix} \quad \text{and} \quad V = V'T^n = \begin{pmatrix} a' & a'n+b' \\ c' & c'n+d' \end{pmatrix}. \tag{1.76}$$

(Note that the results of Lemma 1.5 apply to M' and V'.)

Now since $c = c'$, $\alpha' = \alpha$ and $\gamma' = \gamma$ by (1.59), we have that

$$|c| = |c'| \geq |\alpha'| = |\alpha| \quad \text{and} \quad |c| = |c'| \geq |\gamma'| = |\gamma| \tag{1.77}$$

and (1.72) is proved.

Since the matrices V' and $SM' = \begin{pmatrix} -\gamma' & -\delta' \\ \alpha' & \beta' \end{pmatrix}$ are like the ones in Lemma 1.3 with $q_1, \ldots, q_s$ and $q_1, \ldots, q_{s-1}$ respectively, we have that

$$sgn(c') = sgn(d'), \quad sgn(\alpha') = sgn(\beta') \quad \text{and} \quad sgn(\gamma') = sgn(\delta'). \tag{1.78}$$

Therefore if $n \geq 0$ we have that

$$|c'n+d'| = |c'|n+|d'|, \quad |\alpha'n+\beta'| = |\alpha'|n+|\beta'| \quad \text{and} \quad |\gamma'n+\delta'| = |\gamma'|n+|\delta'|. \tag{1.79}$$

By (1.59) and (1.79), we have that

$$\begin{aligned} |d| = |c'n+d'| &= |c'|n+|d'| \\ &\geq |\alpha'|n+|\beta'| = |\alpha'n+\beta'| = |\beta| \end{aligned} \tag{1.80}$$

and

$$\begin{aligned}|d| = |c'n + d'| &= |c'|n + |d'| \\ &\geq |\gamma'|n + |\delta'| = |\gamma' n + \delta'| = |\delta|.\end{aligned} \tag{1.81}$$

If $n = -1$ then by (1.61) we have that

$$\begin{aligned}|d| = |-c' + d'| &= |c' - d'| \\ &\geq |\alpha' - \beta'| = |-\alpha' + \beta'| = |\beta|\end{aligned} \tag{1.82}$$

and

$$\begin{aligned}|d| = |-c' + d'| &= |c' - d'| \\ &\geq |\gamma' - \delta'| = |-\gamma' + \delta'| = |\delta|.\end{aligned} \tag{1.83}$$

It only remains to show (1.73) for $n < -1$. Note that Lemma 1.4, applied to V' and to SM', implies that

$$|c'| \geq |d'|, \quad |\alpha'| \geq |\beta'| \quad \text{and} \quad |\gamma'| \geq |\delta'|. \tag{1.84}$$

(1.84), (1.78) and the fact that $n < -1$ imply that

$$\begin{aligned}&sgn(-c') = sgn(c'(n+1)) = sgn(-c' + d'), \\ &sgn(-\alpha') = sgn(\alpha'(n+1)) = sgn(-\alpha' + \beta') \quad \text{and} \\ &sgn(-\gamma') = sgn(\gamma'(n+1)) = sgn(-\gamma' + \delta').\end{aligned} \tag{1.85}$$

Thus,

$$\begin{aligned}|d| &= |c'n + d'| \\ &= |c'(n+1) - c' + d'| \\ &= |c'(n+1)| + |-c' + d'| \\ &= |c'||n+1| + |c' - d'|.\end{aligned} \tag{1.86}$$

Similarly,

$$|\delta| = |\gamma'||n+1| + |\gamma' - \delta'| \quad \text{and} \quad |\beta| = |\alpha'||n+1| + |\alpha' - \beta'|. \tag{1.87}$$

(1.86) and (1.87) combined with (1.59) and (1.61) imply that for $n < -1$ we have that

$$\begin{aligned} |d| &= |c'||n+1| + |c'-d'| \\ &\geq |\alpha'||n+1| + |\alpha'-\beta'| = |\beta| \end{aligned} \tag{1.88}$$

and

$$\begin{aligned} |d| &= |c'||n+1| + |c'-d'| \\ &\geq |\gamma'||n+1| + |\gamma'-\delta'| = |\delta|. \end{aligned} \tag{1.89}$$

Corollary 1.1 *Let $q_1, \ldots, q_s \in \mathbb{Z}^+$, $n \in \mathbb{Z}$, and*

$$V = ST^{(-1)^s q_s} ST^{(-1)^{s-1} q_{s-1}} S \ldots T^{q_2} ST^{-q_1} ST^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.90}$$

For $0 \leq j \leq s$ define the matrix

$$M_j = T^{(-1)^j q_j} ST^{(-1)^{j-1} q_{j-1}} S \ldots T^{q_2} ST^{-q_1} ST^n = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}, \tag{1.91}$$

with $M_0 = T^n$ and $SM_s = V$. Then

$$|c| \geq |\alpha_j|, \quad |c| \geq |\gamma_j|, \quad |d| \geq |\beta_j| \quad \text{and} \quad |d| \geq |\delta_j|. \tag{1.92}$$

1.4 The Fourier coefficients of a vector-valued modular form of negative weight

In this section we are going to use the method of Rademacher and Zuckerman [19] to calculate the Fourier coefficients of vector-valued modular forms of negative weight $< -2\delta$, with δ defined in (1.42).

Lemma 1.7 *Let (F, ρ) be a vector-valued modular form of weight $-k$ and $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, with $c > a \geq 0$ and $c > -d \geq 0$. Then if $z = c\tau + d$, we have*

$$f^{(j)}(e^{2\pi i\tau}) = f^{(j)}(e^{2\pi i\frac{z-d}{c}}) = \sum_{l=1}^{p} \Omega_{c,d,l,j}\Psi_{c,l,j}(z) f^{(l)}(e^{2\pi i\frac{a}{c}}e^{-2\pi i\frac{1}{cz}}), \tag{1.93}$$

where $f(x) = \left(f^{(1)}(x), \ldots, f^{(p)}(x)\right)^t$ *is given by (1.16),*

$$\Psi_{c,l,j}(z) = z^k e^{-2\pi i\frac{m_j z + m_l/z}{c}}, \tag{1.94}$$

$$\rho(V)^{-1} = (x^{(j,l)}), \tag{1.95}$$

$$\Omega_{c,d} = v^{-1}(V)\begin{pmatrix} e^{2\pi i m_1\frac{d}{c}} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{2\pi i m_p\frac{d}{c}} \end{pmatrix}\rho(V)^{-1}\begin{pmatrix} e^{2\pi i m_1\frac{a}{c}} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{2\pi i m_p\frac{a}{c}} \end{pmatrix}, \tag{1.96}$$

and we have put

$$\Omega_{c,d,l,j} = (\Omega_{c,d})^{(j,l)} = v^{-1}(V)x^{(j,l)}e^{2\pi i\frac{m_l a + m_j d}{c}}. \tag{1.97}$$

Proof 1.7

First note that if we let $c > 0$ and choose the unique a such that $0 \leq a < c$, then $V \in \Gamma$ is determined by d and c, and $\Omega_{c,d,l,j}$ depends only on d and c. From the definition of vector-valued modular form (1.1, 1.3), we get that

$$F(\tau) = v(V)^{-1}z^k\rho(V)^{-1}F(V\tau). \tag{1.98}$$

Thus,

$$\begin{aligned}
f(x) &= \begin{pmatrix} f^{(1)}(x) \\ \vdots \\ f^{(p)}(x) \end{pmatrix} \\
&= \begin{pmatrix} e^{-2\pi i m_1 \tau} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{-2\pi i m_p \tau} \end{pmatrix} \begin{pmatrix} F^{(1)}(\tau) \\ \vdots \\ F^{(p)}(\tau) \end{pmatrix} \\
&= \begin{pmatrix} e^{-2\pi i m_1 \tau} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{-2\pi i m_p \tau} \end{pmatrix} v^{-1}(V) z^k \rho(V)^{-1} F(V\tau) \qquad (1.99) \\
&= v^{-1}(V) z^k \begin{pmatrix} e^{-2\pi i m_1 \tau} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{-2\pi i m_p \tau} \end{pmatrix} \rho(V)^{-1} \\
&\quad \times \begin{pmatrix} e^{2\pi i m_1 V\tau} & \cdots & o \\ \vdots & \ddots & \vdots \\ o & \cdots & e^{2\pi i m_p V\tau} \end{pmatrix} f(e^{2\pi i V\tau}).
\end{aligned}$$

Now since $(c\tau + d) = z$, and since $ad \equiv 1 \pmod{c}$,

$$\tau = \frac{z-d}{c}, \quad V\tau = \frac{a}{c} - \frac{1}{cz}. \qquad (1.100)$$

Note that since $c > 0$ then $\Im(z) > 0$. Now it only remains to apply (1.100) in (1.99) to prove the lemma.

Lemma 1.8 *Let (F, ρ) be a vector-valued modular form of weight $-k$. Then the Fourier coefficients $a_m = (a_m^{(1)}, \ldots, a_m^{(p)})^t$ are given by the following formula:*

$$a_m^{(j)} = e^{2\pi N^{-2} m} \sum_{\substack{c,\, d \\ 0 \le -d < c \le N \\ (c,\, d) = 1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} \int_{-\theta'_{c,d}}^{\theta''_{c,d}} \Psi_{c,l,j}(ic(N^{-2} - i\varphi)) f^{(l)} \left(e^{\frac{2\pi i}{c}\left(a + ic^{-1}(N^{-2} - i\varphi)^{-1}\right)} \right) e^{-2\pi i m \varphi} d\varphi, \qquad (1.101)$$

for any positive integer N, where $\theta'_{c,d}$ and $\theta''_{c,d}$ are given by the Farey dissection of the circle $|x| = e^{-2\pi N^{-2}}$, with Farey series of order N.

Proof 1.8

Since the functions $f^{(j)}(x)$ are analytic inside the unit circle except possibly at zero where there could be a pole, we can use the Cauchy formula to get

$$a_m^{(j)} = \frac{1}{2\pi i} \int_{C(N)} \frac{f^{(j)}(x)}{x^{m+1}} dx, \tag{1.102}$$

where C is the circle $|x| = e^{2\pi N^{-2}}$, for N a positive integer. We can change the path of integration by making the usual dissection of the circle C into arcs $\xi_{c,d}(N)$, using the Farey series of order N. Thus we have

$$a_m^{(j)} = \sum_{\substack{c,\, d \\ 0 \leq -d < c \leq N \\ (c,\, d) = 1}} \frac{1}{2\pi i} \int_{\xi_{c,d}(N)} \frac{f^{(j)}(x)}{x^{m+1}} dx. \tag{1.103}$$

We can make the change of variable

$$x = e^{-2\pi N^{-2} - 2\pi i \frac{d}{c} + 2i\pi\varphi}, \quad -\theta'_{c,d} \leq \varphi \leq \theta''_{c,d} \tag{1.104}$$

and substitute in (1.103) to get

$$a_m^{(j)} = e^{2\pi N^{-2} m} \sum_{\substack{c,\, d \\ 0 \leq -d < c \leq N \\ (c,\, d) = 1}} e^{2\pi i m \frac{d}{c}} \int_{-\theta'_{c,d}}^{\theta''_{c,d}} f^{(j)}\left(e^{2\pi i \left(i(N^{-2} - i\varphi) - \frac{d}{c}\right)}\right) e^{-2\pi i m\varphi} d\varphi. \tag{1.105}$$

Now we apply (1.93) to $f^{(j)}\left(e^{2\pi i \left(i(N^{-2} - i\varphi) - \frac{d}{c}\right)}\right)$, where $z = ic(N^{-2} - i\varphi)$ and the lemma is proven.

Theorem 1.9 *Let $F(\tau)$ be a vector-valued modular form of weight $-k$ with $k > 2\delta > 0$. (See (1.42) for the definition of δ.) Then for $m \geq 0$ the coefficients in (1.2) are given by the formula*

$$a_m^{(j)} = 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} \sum_{l=1}^{p} \sum_{\nu<0} a_\nu^{(l)} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l}. \tag{1.106}$$

Here,

$$A_{c,\nu,m,j,l} = \sum_{\substack{c,\, d \\ 0 \le -d < c \\ (c,\, d) = 1}} \Omega_{c,d,l,j} e^{2\pi i \frac{dm + a\nu}{c}}, \tag{1.107}$$

$\Omega_{c,d,l,j}$ *is given by (1.97) and* $B_{c,\nu,m,j,l}$ *is given by*

$$B_{c,\nu,m,j,l} = \begin{cases} \left(\frac{-\nu - m_l}{m + m_j}\right)^{\frac{k+1}{2}} I_{k+1}\left(\frac{4\pi}{c}(-\nu - m_l)^{\frac{1}{2}}(m + m_j)^{\frac{1}{2}}\right), & m + m_j > 0 \\ \frac{1}{(k+1)!}\left(\frac{2\pi(-\nu - m_l)}{c}\right)^{k+1}, & m = m_j = 0\ . \end{cases} \tag{1.108}$$

In (1.108),

$$\begin{aligned} I_{k+1}(z) &= \frac{\left(\frac{z}{2}\right)^{k+1}}{2\pi i} \int_{-\infty}^{(0+)} t^{-k} e^{t + \frac{z^2}{4t}} dt, \quad z \in \mathbb{R} \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2n+k+1}}{n!(n+k+1)!}, \quad z \in \mathbb{R}. \end{aligned} \tag{1.109}$$

Proof 1.9

We note first that $f(x)$ in the neighborhood of $x = 0$ is dominated by the principal part $P(x)$, where $P(x)$ and $D(x)$ are the column vectors with components

$$P^{(j)}(x) = \sum_{\nu < 0} a_\nu^{(j)} x^\nu \tag{1.110}$$

and

$$D^{(j)}(x) = \sum_{\nu \ge 0} a_\nu^{(j)} x^\nu, \tag{1.111}$$

so that

$$f^{(j)}(x) = P^{(j)}(x) + D^{(j)}(x). \tag{1.112}$$

For that purpose we split the formula (1.101) for $a_m^{(j)}$ into two parts

$$a_m^{(j)} = Q_m^{(j)}(N) + R_m^{(j)}(N), \tag{1.113}$$

where

$$Q_m^{(j)}(N) = e^{2\pi N^{-2}m} \sum_{\substack{c,\, d \\ 0 \le -d < c \le N \\ (c,\, d) = 1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} \int_{-\theta'_{c,d}}^{\theta''_{c,d}} \Psi_{c,l,j}\left(ic(N^{-2} - i\varphi)\right) P^{(l)}\left(e^{\frac{2\pi i}{c}\left(a + ic^{-1}(N^{-2} - i\varphi)^{-1}\right)}\right) e^{-2\pi i m\varphi} d\varphi \tag{1.114}$$

and

$$R_m^{(j)}(N) = e^{2\pi N^{-2}m} \sum_{\substack{c,\, d \\ 0 \le -d < c \le N \\ (c,\, d) = 1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} \int_{-\theta'_{c,d}}^{\theta''_{c,d}} \Psi_{c,l,j}\left(ic(N^{-2} - i\varphi)\right) D^{(l)}\left(e^{\frac{2\pi i}{c}\left(a + ic^{-1}(N^{-2} - i\varphi)^{-1}\right)}\right) e^{-2\pi i m\varphi} d\varphi. \tag{1.115}$$

We will first show that $\lim_{N\to\infty} R_m^{(j)}(N) = 0$. From the theory of Farey fractions we have

$$\frac{1}{2cN} \le \theta'_{c,d} \le \frac{1}{cN}, \qquad \frac{1}{2cN} \le \theta''_{c,d} \le \frac{1}{cN} \tag{1.116}$$

and we find for $-\theta'_{c,d} \le \varphi \le \theta''_{c,d}$

$$\Re\left(c(N^{-2} - i\varphi)\right) = cN^{-2}. \tag{1.117}$$

By (1.116) and $-\theta'_{c,d} \le \varphi \le \theta''_{c,d}$ we get

$$\begin{aligned} \Re\left(\frac{1}{c(N^{-2} - i\varphi)}\right) &= \frac{N^{-2}}{c(N^{-4} + \varphi^2)} \\ &\ge \frac{N^{-2}}{c(N^{-4} + c^{-2}N^{-2})} \\ &= \frac{c}{c^2N^{-2} + 1} \\ &\ge \frac{c}{2}, \end{aligned} \tag{1.118}$$

since $c \le N$. Also,

$$\begin{aligned} |c(N^{-2} - i\varphi)| &= c(N^{-4} + \varphi^2)^{\frac{1}{2}} \\ &\le (c^2N^{-4} + N^{-2})^{\frac{1}{2}} \\ &\le 2^{\frac{1}{2}} N^{-1}, \end{aligned} \tag{1.119}$$

and by (1.118) and (1.117), we have that

$$\left| e^{-\frac{2\pi i}{c}\left(ic(N^{-2}-i\varphi)m_j+\frac{m_l}{ic(N^{-2}-i\varphi)}\right)} \right| \le e^{\frac{2\pi}{c}\left(cN^{-2}m_j-\frac{cm_l}{2}\right)} \le e^{2\pi\kappa_M N^{-2}} e^{-\pi\kappa_m}, \tag{1.120}$$

where $\kappa_M = max(m_1, \ldots, m_p)$ and $\kappa_m = min(m_1, \ldots, m_p)$. Therefore by (1.119) and (1.120) we have

$$|\Psi_{c,l,j}(z_0)| \le 2^{\frac{k}{2}} N^{-k} e^{2\pi\kappa_M N^{-2}} e^{-\pi\kappa_m},$$

where $z_0 = ic(N^{-2} - i\varphi)$. Also,

$$\left| D^{(l)}\left(e^{\frac{2\pi i}{c}\left(a+ic^{-1}(N^{-2}-i\varphi)^{-1}\right)}\right)\right| \le \sum_{\nu=0}^{\infty} |a_\nu^{(l)}| e^{\frac{-2\pi\nu}{c}\Re\left(c^{-1}(N^{-2}+\varphi)^{-1}\right)} \le \sum_{\nu=0}^{\infty} |a_\nu^{(l)}| e^{-\pi\nu}. \tag{1.121}$$

Using these results we have

$$\begin{aligned} &\left|\Psi_{c.l,j}(ic(N^{-2} - i\varphi))D^{(l)}\left(e^{\frac{2\pi i}{c}\left(a+ic^{-1}(N^{-2}-i\varphi)^{-1}\right)}\right)\right| \\ &\le 2^{\frac{k}{2}} N^{-k} e^{2\pi\kappa_M N^{-2}} e^{-\pi\kappa_m} \sum_{\nu=0}^{\infty} |a_\nu^{(l)}| e^{-\pi\nu} \\ &= CN^{-k} e^{2\pi\kappa_M N^{-2}}. \end{aligned}$$

Here

$$C = 2^{\frac{k}{2}} e^{-\pi\kappa_m} \sum_{\nu=0}^{\infty} |a_\nu^{(l)}| e^{-\pi\nu}, \tag{1.122}$$

which is finite since $|e^{-\pi}| < 1$ and the series is convergent inside the unit circle.

Now in order to bound $|\Omega_{c,d,l,j}|$, we need to use the estimate for ρ discussed before (1.42). For $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ where $c > a \ge 0, c > -d \ge 0$ we have by (1.15) that

$$|\left(\rho^{-1}\right)^{(j,l)}(V)| \le K_3 c^{2\delta} \le K_3 N^{2\delta}, \tag{1.123}$$

where δ, K_2 and K_3 are constants independent of V. Therefore

$$\begin{aligned} |\Omega_{c,d,l,j}| &= \left| v^{-1}(V) x^{(j,l)} e^{2\pi i \frac{m_l a + m_j d}{c}} \right| \\ &\leq K_3 N^{2\delta}. \end{aligned} \tag{1.124}$$

Thus we have,

$$\begin{aligned} &|R_m^{(j)}(N)| \\ &\leq e^{-2\pi N^{-2} m} \sum_{\substack{c,\, d \\ 0 \leq -d < c \leq N \\ (c,\, d) = 1}} \sum_{l=1}^{p} K_3 N^{2\delta} \int_{-\theta'_{c,d}}^{\theta''_{c,d}} C N^{-k} e^{2\pi\delta N^{-2}} d\varphi \\ &\leq K_4 e^{-2\pi N^{-2}(m-\delta)} N^{-k+2\delta} \sum_{\substack{c,\, d \\ 0 \leq -d < c \leq N \\ (c,\, d) = 1}} \int_{-\theta'_{c,d}}^{\theta''_{c,d}} d\varphi \\ &= K_4 e^{-2\pi N^{-2}(m-\delta)} N^{-k+2\delta}\ , \end{aligned} \tag{1.125}$$

where K_4 is a constant independent of V. We conclude that $\lim_{N\to\infty} |R_m^{(j)}(N)| = 0$ for $k > 2\delta$.

Lemma 1.10 *The following series converges absolutely for $k > 2\delta$:*

$$\sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l} B_{c,\nu,m,j,l}, \tag{1.126}$$

where $A_{c,\nu,m,j,l}$ is given by (1.107) and $B_{c,\nu,m,j,l}$ is given by (1.108).

Proof 1.10

From (1.123) and (1.107) we have that

$$\left| \frac{i^k}{c} A_{c,\nu,m,j,l} \right| \leq K_4 c^{2\delta}. \tag{1.127}$$

On the other hand, for $m + m_j > 0$ we have that

$$\begin{aligned}
&B_{c,\nu,m,j,l} \\
&= \left(\frac{-\nu - m_l}{m + m_j}\right)^{\frac{k+1}{2}} I_{k+1}\left(\frac{4\pi}{c}(-\nu - m_l)^{\frac{1}{2}}(m + m_j)^{\frac{1}{2}}\right) \\
&= \left(\frac{-\nu - m_l}{m + m_j}\right)^{\frac{k+1}{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{4\pi}{c}(-\nu - m_l)^{\frac{1}{2}}(m + m_j)^{\frac{1}{2}}\right)^{2n+k+1}}{2^{2n+k+1} n!(n + k + 1)!} \\
&= \left(\frac{-\nu - m_l}{c}\right)^{k+1} \sum_{n=0}^{\infty} \left(\frac{4\pi^2(-\nu - m_l)(m + m_j)}{c^2}\right)^{n} \frac{1}{n!(n + k + 1)!} \\
&\leq \left(\frac{-\nu - m_l}{c}\right)^{k+1} e^{4\pi^2(-\nu - m_l)(m + m_j)},
\end{aligned} \tag{1.128}$$

and for $m = m_j = 0$, we have that

$$B_{c,\nu,m,j,l} = \frac{1}{(k+1)!}\left(\frac{2\pi(-\nu - m_l)}{c}\right)^{k+1}. \tag{1.129}$$

In both cases the series (1.126) converge absolutely for $k > 2\delta$.

Lemma 1.11 *Let $F(\tau)$ be a vector-valued modular form of weight $-k$ with $k > 2\delta > 0$. Then if $m + m_j > 0$ the Fourier coefficients in (1.2) are given by the formula*

$$a_m^{(j)} = \sum_{c=1}^{\infty} (ic)^k \sum_{l=1}^{p} \sum_{\nu<0} a_\nu^{(l)} A_{c,\nu,m,j,l} \left(L_{c,\nu,m,j,l} + H_{c,\nu,m,j,l}\right). \tag{1.130}$$

Here $A_{c,\nu,m,j,l}$ is given by (1.107),

$$L_{c,\nu,m,l}^{(j)} = \frac{1}{i}\int_{-\infty}^{(0+)} g^{(j)}(\omega, c, \nu, l)\,d\omega, \tag{1.131}$$

and

$$H_{c,\nu,m,j,l} = 2\sin\pi k \int_{0}^{\infty} g^{(j)}(\omega, c, \nu, l)\,d\omega, \tag{1.132}$$

where

$$g^{(j)}(\omega, c, \nu, l) = \omega^k e^{-\frac{2\pi(\nu + m_l)}{c^2\omega}} e^{2\pi(m + m_j)\omega}, \tag{1.133}$$

and $\int_{-\infty}^{(0+)}$ *is the integral on the keyhole path, which is the path that goes from* $-\infty$ *to 0, with the argument* π*, and then goes from 0 to* $-\infty$ *with the argument* $-\pi$*.*

Proof 1.11

We evaluate $Q_m(N)$, under the above conditions. If we substitute $\omega = N^{-2} - i\varphi$ in (1.114), we have

$$Q_m^{(j)}(N) = \sum_{\substack{c,\,d \\ 0 \le -d < c \le N \\ (c,\,d)=1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} \frac{1}{i} \int_{N^{-2}-\theta''_{c,d}}^{N^{-2}+\theta'_{c,d}} \Psi_{c,l,j}(ic\omega) \times P^{(l)} \left(e^{\frac{2\pi i}{c}\left(a + ic^{-1}\omega^{-1}\right)} \right) e^{2\pi m\omega} d\omega. \tag{1.134}$$

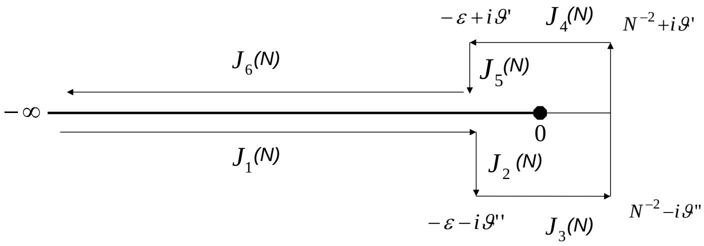
Therefore by (1.94) and (1.110), we have

$$\begin{aligned} Q_m^{(j)}(N) &= \sum_{\substack{c,\,d \\ 0 \le -d < c \le N \\ (c,\,d)=1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} \frac{1}{i} \int_{N^{-2}-\theta''_{c,d}}^{N^{-2}+\theta'_{c,d}} (ic)^k \omega^k e^{2\pi m_j \omega} e^{-\frac{2\pi m_l}{c^2\omega}} \\ &\qquad \times \sum_{\nu<0} a_\nu^{(l)} e^{\frac{2\pi i a\nu}{c}} e^{-\frac{2\pi\nu}{c^2\omega}} e^{2\pi m\omega} d\omega \qquad (1.135) \\ &= \sum_{\substack{c,\,d \\ 0 \le -d < c \le N \\ (c,\,d)=1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} (ic)^k \sum_{\nu<0} e^{\frac{2\pi i a\nu}{c}} a_\nu^{(l)} I_{c,\nu,m,j,l}\ , \end{aligned}$$

where

$$\begin{aligned} I_{c,\nu,m,j,l} &= \frac{1}{i} \int_{N^{-2}-i\theta''_{c,d}}^{N^{-2}+i\theta'_{c,d}} \omega^k e^{-\frac{2\pi(\nu+m_l)}{c^2\omega}} e^{2\pi(m+m_j)\omega} d\omega \\ &= \frac{1}{i} \int_{N^{-2}-i\theta''_{c,d}}^{N^{-2}+i\theta'_{c,d}} g^{(j)}(\omega, c, \nu, l) d\omega. \end{aligned}$$

Now we cut the complex plane from 0 to $-\infty$ along the negative real axis, and consider the path shown in the figure below, with ε, θ' and $\theta'' > 0$.



Then we can write

$$\begin{aligned}I_{c,\nu,m,j,l} = \frac{1}{i}\int_{-\infty}^{(0+)} - \frac{1}{i}\int_{-\infty}^{-\varepsilon} - \frac{1}{i}\int_{-\varepsilon}^{-\varepsilon-i\theta''_{c,d}} - \frac{1}{i}\int_{-\varepsilon-i\theta''_{c,d}}^{N^{-2}-i\theta''_{c,d}} \\ - \frac{1}{i}\int_{N^{-2}+i\theta'_{c,d}}^{-\varepsilon+i\theta'_{c,d}} - \frac{1}{i}\int_{-\varepsilon+i\theta'_{c,d}}^{-\varepsilon} - \frac{1}{i}\int_{-\varepsilon}^{-\infty} \\ = L_{c,\nu,m,j,l} - J_1(N) - J_2(N) - J_3(N) - J_4(N) - J_5(N) - J_6(N),\end{aligned} \tag{1.136}$$

where the integrand in all the integrals is

$$\omega^k e^{-\frac{2\pi(\nu+m_l)}{c^2\omega}} e^{2\pi(m+m_j)\omega}. \tag{1.137}$$

Note that the argument in $J_1(N)$ is $-\pi$ and the argument in $J_6(N)$ is π.

We also assume that $0 < \varepsilon < N^{-2}$. Now in the integral $J_2(N)$ we have

$$\begin{aligned}\omega &= -\varepsilon + i\upsilon, & 0 \geq \upsilon \geq -\theta''_{c,d}, \\ \Re(\omega) &= -\varepsilon, & \Re\left(\tfrac{1}{\omega}\right) = \tfrac{-\varepsilon}{\varepsilon^2+\upsilon^2} < 0, \\ \text{and} \quad |\omega| &= (\varepsilon^2+\upsilon^2)^{\frac{1}{2}} \leq (N^{-4}+c^{-2}N^{-2})^{\frac{1}{2}} \leq 2^{\frac{1}{2}}c^{-1}N^{-1}. \end{aligned} \tag{1.138}$$

Therefore

$$\begin{aligned}|J_2(N)| &\leq \theta''_{c,d} 2^{\frac{k}{2}} c^{-k} N^{-k} e^{-2\pi(m+m_j)\varepsilon} \\ &< 2^{\frac{k}{2}} c^{-k-1} N^{-k-1}.\end{aligned} \tag{1.139}$$

Similarly we have

$$|J_5(N)| < 2^{\frac{k}{2}} c^{-k-1} N^{-k-1}. \tag{1.140}$$

In the integral $J_3(N)$, we have

$$\begin{aligned} \omega &= -u - i\theta''_{c,d}, & -N^{-2} \le -\varepsilon \le u \le N^{-2}, \\ \Re(\omega) &= u \le N^{-2}, & \Re\left(\tfrac{1}{\omega}\right) = \tfrac{u}{u^2+\theta''^2_{c,d}} \le \tfrac{N^{-2}}{\theta''^2_{c,d}} \le 4c^2, \\ |\omega| &= \left(u^2 + \theta''^2_{c,d}\right)^{\frac{1}{2}} \le \left(N^{-4} + c^{-2}N^{-2}\right)^{\frac{1}{2}} \le 2^{\frac{1}{2}} c^{-1} N^{1}, \end{aligned} \tag{1.141}$$

and therefore,

$$\begin{aligned} |J_3(N)| &\le \left(N^{-2} + \varepsilon\right) 2^{\frac{k}{2}} c^{-k} N^{-k} e^{2\pi(m+m_j)N^{-2} - 8\pi(\nu+m_l)} \\ &\le 2^{1+\frac{k}{2}} c^{-k-1} N^{-k-1} e^{2\pi(m+m_j)N^{-2} - 8\pi(\nu+m_l)}. \end{aligned} \tag{1.142}$$

Similarly,

$$|J_4(N)| \le 2^{1+\frac{k}{2}} c^{-k-1} N^{-k-1} e^{2\pi(m+m_j)N^{-2} - 8\pi(\nu+m_l)}. \tag{1.143}$$

Finally, we have

$$J_1(N) + J_6(N) = \frac{e^{-\pi i k}}{i} \int_{-\infty}^{-\varepsilon} + \frac{e^{\pi i k}}{i} \int_{-\varepsilon}^{-\infty}, \tag{1.144}$$

where the integrand is given by

$$|\omega|^k e^{-\frac{2\pi(\nu+m_l)}{c^2\omega}} e^{2\pi(m+m_j)\omega}. \tag{1.145}$$

Thus we get

$$J_1(N,\varepsilon) + J_6(N,\varepsilon) = -2\sin\pi k \int_{\varepsilon}^{\infty} t^k e^{\frac{2\pi(m_l+\nu)}{c^2 t}} e^{-2\pi(m+m_j)t} dt. \tag{1.146}$$

(Here, for clarity we have written $J_1(N) = J_1(N,\varepsilon)$ and $J_6(N) = J_6(N,\varepsilon)$.) Combining (1.136), (1.139), (1.140), (1.142), (1.143), (1.146) and making $\varepsilon \to 0+$, we get

$$I_{c,\nu,m,j,l} = L_{c,\nu,m,j,l} + H_{c,\nu,m,j,l} + 6\Theta_1 2^{\frac{k}{2}} c^{-k-1} N^{-k-1} e^{2\pi(m+m_j)N^{-2} - 8\pi(\mu+m_l)}, \tag{1.147}$$

where μ is the smallest ν such that $a_\mu(h) \neq 0$ for some h, $|\Theta_1| < 1$,

$$L_{c,\nu,m,j,l} = \frac{1}{i}\int_{-\infty}^{(0+)} \omega^k e^{-\frac{2\pi(\nu+m_l)}{c^2\omega}} e^{2\pi(m+m_j)\omega} d\omega \tag{1.148}$$

and

$$H_{c,\nu,m,j,l} = 2\sin \pi k \int_{\varepsilon}^{\infty} t^k e^{\frac{2\pi(m_l+\nu)}{c^2 t}} e^{-2\pi(m+m_j)t} dt. \tag{1.149}$$

Now by (1.135) , (1.124), the fact that $\sum_1^N c^{-1} \leq N$ and (1.123), we obtain

$$\begin{aligned}
Q_m^{(j)}(N) = & \sum_{\substack{c,\,d \\ 0 \leq -d < c \leq N \\ (c,\,d)=1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} (ic)^k \\
& \times \sum_{\nu<0} e^{\frac{2\pi i a\nu}{c}} a_\nu^{(l)} (L_{c,\nu,m,j,l} + H_{c,\nu,m,j,l}) \\
& + \sum_{\substack{c,\,d \\ 0 \leq -d < c \leq N \\ (c,\,d)=1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} (ic)^k \\
& \times \sum_{\nu<0} e^{\frac{2\pi i a\nu}{c}} a_\nu^{(l)} 6\Theta_1 2^{\frac{k}{2}} c^{-k-1} N^{-k-1} e^{2\pi(m+m_j)N^{-2} - 8\pi(\mu+m_l)} \\
= & \sum_{\substack{c,\,d \\ 0 \leq -d < c \leq N \\ (c,\,d)=1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} (ic)^k \\
& \times \sum_{\nu<0} e^{\frac{2\pi i a\nu}{c}} a_\nu^{(l)} (L_{c,\nu,m,j,l} + H_{c,\nu,m,j,l}) \\
& + \sum_{\nu<0} |a_\nu^{(l)}| 6\Theta_2 2^{\frac{k}{2}} N^{-k-1} e^{2\pi(m+m_j)N^{-2} - 8\pi(\mu+m_l)} \sum_{\substack{c,\,d \\ 0 \leq -d < c \leq N \\ (c,\,d)=1}} x^{(j,l)} c^{-1} \\
= & \sum_{\substack{c,\,d \\ 0 \leq -d < c \leq N \\ (c,\,d)=1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} (ic)^k \\
& \times \sum_{\nu<0} e^{\frac{2\pi i a\nu}{c}} a_\nu^{(l)} (L_{c,\nu,m,j,l} + H_{c,\nu,m,j,l}) \\
& + O(N^{-k+2\delta} e^{2\pi(m+m_j)N^{-2}}),
\end{aligned} \tag{1.150}$$

where $|\Theta_2| < 1$ and $|\Theta_1| < 1$. Thus by (1.150), (1.113) and (1.125), we have

$$\begin{aligned} a_m^{(j)} = \sum_{c=1}^{N} (ic)^k \sum_{\substack{0 \le -d < c \\ (c,d)=1}} \sum_{l=1}^{p} \Omega_{c,d,l,j} e^{2\pi i d \frac{m}{c}} \\ \times \sum_{\nu<0} e^{\frac{2\pi i a \nu}{c}} a_\nu^{(l)} (L_{c,\nu,m,j,l} + H_{c,\nu,m,j,l}) \\ + O(N^{-k+2\delta} e^{2\pi(m+m_j)N^{-2}}). \end{aligned} \tag{1.151}$$

Now let $N \to \infty$. Since $k > 2\delta$, by Lemma 1.10 we have that the series converges absolutely and therefore

$$a_m^{(j)} = \sum_{c=1}^{\infty} (ic)^k \sum_{l=1}^{p} \sum_{\nu<0} a_\nu^{(l)} A_{c,\nu,m,j,l} \left(L_{c,\nu,m,j,l} + H_{c,\nu,m,j,l} \right), \tag{1.152}$$

where $A_{c,\nu,m,j,l}$ is given by (1.107). Now from the theory of Bessel functions we have that

$$\begin{aligned} L_{c,\nu,m,j,l} + H_{c,\nu,m,j,l} &= \frac{2\pi}{c^{k+1}} \left(\frac{-\nu - m_l}{m + m_j} \right)^{\frac{k+1}{2}} I_{k+1} \left(\frac{4\pi}{c} (-\nu - m_l)^{\frac{1}{2}} (m + m_j)^{\frac{1}{2}} \right) \\ &= \frac{2\pi}{c^{k+1}} B_{c,\nu,m,j,l}, \end{aligned} \tag{1.153}$$

well defined since $m + m_j > 0$. Here $I_{k+1}(z)$ is given by (1.109). This reduces (1.152) to

$$a_m^{(j)} = 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} \sum_{l=1}^{p} \sum_{\nu<0} a_\nu^{(l)} A_{c,\nu,m,j,l} \ B_{c,\nu,m,j,l}, \tag{1.154}$$

where $B_{c,\nu,m,j,l}$ is given by (1.108). This completes the proof of Theorem 1.9 for the case $m + m_j > 0$.

In order to complete the proof of Theorem 1.9 for the case $m = m_j = 0$, we note that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{m,m_j\to 0}\{L_{c,\nu,m,j,l} + J_1(N,\varepsilon) + J_6(N,\varepsilon)\} \\
&= \lim_{m,m_j\to 0}\lim_{\varepsilon\to 0}\{L_{c,\nu,m,j,l} + J_1(N,\varepsilon) + J_6(N,\varepsilon)\} \\
&= \lim_{m,m_j\to 0}\{L_{c,\nu,m,j,l} + H_{c,\nu,m,j,l}\} \\
&= \lim_{m,m_j\to 0}\frac{2\pi}{c^{k+1}}\left(\frac{-\nu - m_l}{m+m_j}\right)^{\frac{k+1}{2}} I_{k+1}\left(\frac{4\pi}{c}(-\nu - m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right) \\
&= \frac{2\pi}{(k+1)!}\left(\frac{2\pi(-\nu - m_l)}{c^2}\right)^{k+1}.
\end{aligned} \tag{1.155}$$

Also we note that the estimates (1.139), (1.140), (1.142) and (1.143) hold. Therefore,

Lemma 1.12 *Let $F(\tau)$ be a vector-valued modular form of weight $-k$, with $k > 2\delta > 0$. Then if $m_j = 0$,*

$$a_0^{(j)} = 2\pi\sum_{c=1}^{\infty}\frac{i^k}{c}\sum_{l=1}^{p}\sum_{\nu<0} a_\nu^{(l)} A_{c,\nu,0,j,l}\ B_{c,\nu,0,j,l} \tag{1.156}$$

where $A_{c,\nu,0,j,l}$ is given by (1.107) when $m = 0$, and

$$B_{c,\nu,0,j,l} = \frac{1}{(k+1)!}\left(\frac{2\pi(-\nu - m_l)}{c}\right)^{k+1}. \tag{1.157}$$

Theorem 1.9 follows from (1.154) and Lemma 1.12.

CHAPTER 2

Construction of Vector Valued Modular Forms of negative weight

2.1 Introduction

In the first chapter we saw that the Fourier coefficients of vector-valued modular forms (1.2) of negative weight are given by (1.106). Now the question is: whether, given a set of column vectors $b_{-1}, \ldots, b_{\mu}$, a representation ρ on $\Gamma(1)$ and a multiplier system v on $\Gamma(1)$ in weight $-k$ for $k > 2\delta$ (1.42) and $k \in \mathbb{Z}$, the formula (1.106) gives rise to a vector-valued modular form of negative weight $-k$. The answer: not necessarily. The transformation law (1.1) does not necessarily hold, although it does hold up to an aditive polynomial of degree at most k. This is the content of our next theorem.

Theorem 2.1 *Let $b_{-1}, \ldots, b_{\mu}$ be a set of column vectors such that $b_{\nu} \in \mathbb{C}^p$, $b_{\mu} \neq 0$, $\rho : \Gamma(1) \longrightarrow GL(p, \mathbb{C})$ a p-dimensional complex representation , v a multiplier system on $\Gamma(1)$ and weight $-k$, and $k > 2\delta$ (see 1.42), with*

$k, -\mu \in \mathbb{Z}^+$. Define

$$F(\tau) = \begin{pmatrix} \sum_{\mu \le \nu < 0} b_\nu^{(1)} e^{2\pi i(m_1+\nu)\tau} + \sum_{m=0}^\infty b_m^{(1)} e^{2\pi i(m+m_1)\tau} \\ \vdots \\ \sum_{\mu \le \nu < 0} b_\nu^{(p)} e^{2\pi i(m_p+\nu)\tau} + \sum_{m=0}^\infty b_m^{(p)} e^{2\pi i(m+m_p)\tau} \end{pmatrix}, \tag{2.1}$$

where for $m \ge 0$ we define b_m as the column vector with components given by

$$b_m^{(j)} = 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} \sum_{l=1}^{p} \sum_{\mu \le \nu < 0} b_\nu^{(l)} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l}. \tag{2.2}$$

with $A_{c,\nu,m,j,l}$ given by (1.107) and $B_{c,\nu,m,j,l}$ by (1.108).

Then

1. *$F(\tau)$ is regular in the complex upper half-plane $\mathcal{H}$, and*

2. *$F(\tau)$ satisfies*

$$F(\tau) - \upsilon^{-1}(M)(\gamma\tau+\delta)^k \rho^{-1}(M) F(M\tau) = Q_M(\tau, k, \upsilon, \rho), \tag{2.3}$$

for all

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma, \tag{2.4}$$

where $Q_M(\tau, k, \upsilon, \rho)$ is a column vector of polynomials in τ of degree at most k.

Lemma 2.2 *For $k > 2\delta$ the series*

$$2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} \tag{2.5}$$

converges absolutely, and as $m \to \infty$, we have

$$2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} = O\left((m+\kappa_m)^{-\frac{3}{4}-\frac{k}{2}} e^{4\pi(\nu-\kappa_m)^{\frac{1}{2}}(m+\kappa_M)^{\frac{1}{2}}} \right), \tag{2.6}$$

where

$$\kappa_m = \min(m_1, \ldots, m_p) \quad \textit{and} \quad \kappa_M = \max(m_1, \ldots, m_p). \tag{2.7}$$

Proof 2.2

The strategy is the same as in [5]. First we will show that

$$\left| 2\pi \sum_{c=2}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} \right| \leq C(m+m_j)^{-\frac{1}{2}} e^{2\pi (m+m_j)^{\frac{1}{2}}(-\nu-m_l)^{\frac{1}{2}}}. \tag{2.8}$$

Then we show that as $m \to \infty$ the summation on c is dominated by the term for $c = 1$.

In order to bound $A_{c,\nu,m,j,l}$, we use (1.123) to get

$$\begin{aligned} |A_{c,\nu,m,j,l}| &= \left| \sum_{\substack{c,\, d \\ 0 \leq -d < c \\ (c,\, d) = 1}} \Omega_{c,d,l,j} e^{2\pi i \frac{dm+a\nu}{c}} \right| \\ &= \left| \sum_{\substack{c,\, d \\ 0 \leq -d < c \\ (c,\, d) = 1}} v^{-1}(V) x^{(j,l)} e^{2\pi i \frac{m_l a + m_j d}{c}} e^{2\pi i \frac{dm+a\nu}{c}} \right| \\ &= O\left(c^{2\delta+1}\right). \end{aligned} \tag{2.9}$$

On the other hand, from the power series definition of $I_{k+1}(z)$ (1.109) we have that

$$I_{k+1}(z) \leq z^k \sinh z. \tag{2.10}$$

Also, we have

$$\sinh z \leq \frac{z}{B} \sinh B, \quad \text{for } 0 \leq z \leq B. \tag{2.11}$$

Now by (2.9), (2.10), (2.11), (1.108) and the fact that $k > 2\delta$, we have, for $m + m_j > 0$,

$$\begin{aligned}
&\left| 2\pi \sum_{c=2}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} \right| \\
&\quad \le 2\pi \sum_{c=2}^{\infty} \frac{1}{c} \left| \sum_{\substack{c,\,d \\ 0 \le -d < c \\ (c,\,d) = 1}} v^{-1}(V) x^{(j,l)} e^{2\pi i \frac{m_l a + m_j d}{c}} e^{2\pi i \frac{dm + a\nu}{c}} B_{c,\nu,m,j,l} \right| \\
&\quad \le C_1 \frac{(-\nu - m_l)^{k+\frac{1}{2}}}{(m + m_j)^{\frac{1}{2}}} \sinh\left(2\pi (m + m_j)^{\frac{1}{2}} (-\nu - m_l)^{\frac{1}{2}} \right) \sum_{c=2}^{\infty} c^{-k+2\delta-1} \\
&\quad \le C_2 (m + m_j)^{-\frac{1}{2}} e^{2\pi (m+m_j)^{\frac{1}{2}} (-\nu - m_l)^{\frac{1}{2}}}.
\end{aligned} \tag{2.12}$$

Also from (2.9), (1.108) and the fact that $k > 2\delta$, we have for $m + m_j = 0$ that

$$2\pi \sum_{c=2}^{\infty} \left| \frac{i^k}{c} A_{c,\nu,0}^{(l,j)}\ B_{c,\nu,0,l}^{(j)} \right| \le C_3 \frac{(-\nu - m_l)}{(k+1)!} \sum_{c=2}^{\infty} c^{2\delta - 3}. \tag{2.13}$$

The term for $c = 1$ is

$$\begin{aligned}
&2\pi i^k A_{1,\nu,m,j,l}\ B_{1,\nu,m,l}^{(j)} \\
&= 2\pi v^{-1}(V) x^{(j,l)} e^{2\pi i (m_l a + m_j d)} e^{2\pi i (dm + a\nu)} \left(\frac{-\nu - m_l}{m + m_j} \right)^{\frac{k+1}{2}} I_{k+1}\left(4\pi (-\nu - m_l)^{\frac{1}{2}} (m + m_j)^{\frac{1}{2}} \right).
\end{aligned} \tag{2.14}$$

Also, by [20], we have

$$I_{k+1}(z) \sim \frac{e^z}{\sqrt{2\pi z}}. \tag{2.15}$$

Thus, the behavior for the term $c = 1$ is given by

$$2\pi i^k A_{1,\nu,m,j,l}\ B_{1,\nu,m,j,l} = O\left((m + m_j)^{-\frac{k}{2} - \frac{3}{4}} e^{4\pi (-\nu - m_l)^{\frac{1}{2}} (m + m_j)^{\frac{1}{2}}} \right). \tag{2.16}$$

Thus we see that the series

$$2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} \tag{2.17}$$

converges absolutely, and that

$$2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} = O\left((m+\kappa_m)^{-\frac{3}{4}-\frac{k}{2}} e^{4\pi(-\nu-\kappa_m)^{\frac{1}{2}}(m+\kappa_M)^{\frac{1}{2}}} \right). \quad (2.18)$$

Corollary 2.3 *The series*

$$\sum_{m=0}^{\infty} \left| 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)\tau} \right| \quad (2.19)$$

converges uniformly in τ *on* $I_w = \{\tau : \mathcal{I}(\tau) > w > 0\}$.

Proof 2.3

For $m > 0$ and $\mathcal{I}(\tau) > w$, we have that

$$\begin{aligned} \left| 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)\tau} \right| &\leq C_1 \left| (m+\kappa_m)^{-\frac{3}{4}-\frac{k}{2}} e^{4\pi(-\nu-\kappa_m)^{\frac{1}{2}}(m+\kappa_M)^{\frac{1}{2}}} e^{2\pi i(m+m_j)\tau} \right| \\ &\leq C_2 m^{-\frac{3}{4}-\frac{k}{2}} e^{-2\pi m w + 4\pi\mu^{\frac{1}{2}}(m+1)^{\frac{1}{2}}}. \end{aligned} \quad (2.20)$$

Proof 2.1

Let $\mathcal{R}_\nu(\tau)$ be the matrix function defined by

$$\mathcal{R}_\nu^{(j,l)}(\tau) = \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)\tau}. \quad (2.21)$$

By Corollary 2.3 and Lemma 2.2, we have that $\mathcal{R}_\nu^{(j,l)}(\tau)$ converges absolutely in m and in c, and therefore we can change the order of summation in (2.1) and rewrite the j^{th} component of $F(\tau)$ in the form

$$\begin{aligned} F^{(j)}(\tau) &= \sum_{\mu\leq\nu<0} b_\nu^{(j)} e^{2\pi i(m_j+\nu)\tau} + 2\pi \sum_{m=0}^{\infty}\sum_{c=1}^{\infty} \frac{i^k}{c} \sum_{l=1}^{p} \sum_{\mu\leq\nu<0} b_\nu^{(l)} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)\tau} \\ &= \sum_{\mu\leq\nu<0} b_\nu^{(j)} e^{2\pi i(m_j+\nu)\tau} + \sum_{\mu\leq\nu<0} \sum_{l=1}^{p} b_\nu^{(l)} \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)\tau} \\ &= \sum_{\mu\leq\nu<0} b_\nu^{(j)} e^{2\pi i(m_j+\nu)\tau} + \sum_{\mu\leq\nu<0} \sum_{l=1}^{p} b_\nu^{(l)} \mathcal{R}_\nu^{(j,l)}(\tau). \end{aligned} \quad (2.22)$$

Since the series converges uniformly on compacts of $\mathcal{H}$ by Corollary 2.3, $F(\tau)$ is regular in $\mathcal{H}$. Now by (2.22) we can rewrite the function $F(\tau)$ as

$$F(\tau) = \sum_{\mu \le \nu < 0} \mathcal{T}_\nu(\tau) \begin{pmatrix} b_\nu^{(1)} \\ \vdots \\ b_\nu^{(p)} \end{pmatrix}, \tag{2.23}$$

where $\mathcal{T}_\nu(\tau)$ is the matrix given by

$$\mathcal{T}_\nu^{(j,l)}(\tau) = \delta_{jl} e^{2\pi i(m_j+\nu)\tau} + \mathcal{R}_\nu^{(j,l)}(\tau). \tag{2.24}$$

We will prove the result for $\tau = iy$ and $y > 0$, and by analytic continuation the result will follow for all τ in $\mathcal{H}$.

Now by (2.21), (1.107) and the absolute convergence of the double series $\mathcal{R}_\nu^{(j,l)}$ in m and c we have that

$$\begin{aligned}
\mathcal{R}_\nu^{(j,l)}(\tau) &= \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\ B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)\tau} \\
&= \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} \sum_{\substack{c,\,d \\ 0 \le -d < c \\ (c,\,d)=1}} v^{-1}(V) x^{(j,l)} e^{2\pi i \frac{m_l a + m_j d}{c}} e^{2\pi i \frac{dm + a\nu}{c}}\ B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)\tau} \\
&= 2\pi \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{c,\,d \\ 0 \le -d < c \\ (c,\,d)=1}} v^{-1}(V) x^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)(\tau + \frac{d}{c})}.
\end{aligned} \tag{2.25}$$

To proceed we need the Lipschitz summation formula.

$$\text{For} \quad n > -1, \quad 0 < m_j < 1 \quad \text{and} \quad \mathcal{I}(\tau) > 0, \tag{2.26}$$

$$\sum_{m=0}^{\infty} (m+m_j)^n e^{2\pi i\tau(m+m_j)} = \frac{\Gamma(n+1)}{(2\pi)^{n+1}} \sum_{q=-\infty}^{\infty} e^{2\pi i q m_j} \left(-i\left(\tau - q\right)\right)^{-n-1}. \tag{2.27}$$

$$\text{If} \quad n = 0, \quad m_j = 0 \quad \text{and} \quad \mathcal{I}(\tau) > 0, \tag{2.28}$$

$$\sum_{m=1}^{\infty} (m+m_j)^n e^{2\pi i\tau(m+m_j)} = -\frac{1}{2} + \frac{1}{2\pi} \sum_{q=-\infty}^{\infty}{}^{*} \left(-i\left(\tau - q\right)\right)^{-1}, \tag{2.29}$$

where $\sum_{q=-\infty}^{\infty}{}^{*} = \lim_{N\to\infty} \sum_{q=-N}^{N}$.

Lemma 2.4 *If $m_j > 0$, we have*

$$\frac{2\pi}{c} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l} e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} = \sum_{q=-\infty}^{\infty} e^{2\pi i m_j q} \left(c\tau + d - cq\right)^k \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d - cq)}\right)^n, \tag{2.30}$$

and if $m_j = 0$ we have

$$\begin{aligned} &\frac{2\pi}{c} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l} e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} \\ &= \frac{-\pi i^k}{c} B_{c,\nu,0,j,l} + \sum_{q=-\infty}^{\infty}{}^{*} e^{2\pi i q m_j} \left(c\tau + d - cq\right)^k \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d - cq)}\right)^n. \end{aligned} \tag{2.31}$$

Proof 2.4

The proof is a simple application of the Lipschitz summation formula (2.27), (2.29), the definition of $B_{c,\nu,m,j,l}$ (1.108) and of the power series expansion of $I_{k+1}(z)$ (1.109). For $m + m_j > 0$, we have:

$$\begin{aligned} &\frac{2\pi}{c} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l} e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} \\ &= \frac{2\pi}{c} i^k \sum_{m=0}^{\infty} \left(\frac{-\nu - m_l}{m + m_j}\right)^{\frac{k+1}{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{4\pi}{c}(-\nu - m_l)^{\frac{1}{2}}(m + m_j)^{\frac{1}{2}}\right)^{2n+k+1}}{2^{2n+k+1} n!(n + k + 1)!} e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} \\ &= \frac{2\pi}{c} i^k \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\nu - m_l)^{n+k+1}(m + m_j)^n}{n!(n + k + 1)!} e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} \\ &= \frac{2\pi}{c} i^k \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\nu - m_l)^{n+k+1}}{n!(n + k + 1)!} \sum_{m=0}^{\infty} (m + m_j)^n e^{2\pi i (m+m_j)(\tau+\frac{d}{c})} \\ &= \frac{2\pi}{c} i^k \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\nu - m_l)^{n+k+1}}{n!(n + k + 1)!} \frac{\Gamma(n + 1)}{(2\pi)^{n+1}} \sum_{q=-\infty}^{\infty} e^{2\pi i q m_j} \left(-i\left(\tau + \frac{d}{c} - q\right)\right)^{-n-1} \\ &= \sum_{q=-\infty}^{\infty} e^{2\pi i q m_j} \left(c\tau + d - cq\right)^k \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau + d - cq)}\right)^n. \end{aligned} \tag{2.32}$$

Now for $m_j = 0$, we have that

$$
\begin{aligned}
&\frac{2\pi}{c} i^k \sum_{m=1}^{\infty} B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)(\tau+\frac{d}{c})} \\
&= \frac{2\pi}{c} i^k \sum_{m=1}^{\infty} \left(\frac{-\nu-m_l}{m+m_j}\right)^{\frac{k+1}{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{4\pi}{c}(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right)^{2n+k+1}}{2^{2n+k+1} n!(n+k+1)!} e^{2\pi i(m+m_j)(\tau+\frac{d}{c})} \\
&= \frac{2\pi}{c} i^k \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\nu-m_l)^{n+k+1}(m+m_j)^n}{n!(n+k+1)!} e^{2\pi i(m+m_j)(\tau+\frac{d}{c})} \\
&= \frac{2\pi}{c} i^k \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\nu-m_l)^{n+k+1}}{n!(n+k+1)!} \sum_{m=1}^{\infty} (m+m_j)^n e^{2\pi i(m+m_j)(\tau+\frac{d}{c})} \\
&= \frac{2\pi}{c} i^k \Bigg\{ -\frac{1}{2}\left(\frac{2\pi}{c}\right)^{k+1} \frac{(-\nu-m_l)^{k+1}}{(k+1)!} \\
&\qquad + \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\nu-m_l)^{n+k+1}}{n!(n+k+1)!} \frac{\Gamma(n+1)}{(2\pi)^{n+1}} \sideset{}{^*}\sum_{q=-\infty}^{\infty} e^{2\pi i q m_j} \left(-i\left(\tau+\frac{d}{c}-q\right)\right)^{-n-1} \Bigg\} \\
&= -\frac{\pi i^k}{c} B_{c,\nu,0,j,l} + \sideset{}{^*}\sum_{q=-\infty}^{\infty} e^{2\pi i q m_j} (c\tau+d-cq)^k \sum_{n=k+1}^{\infty} \frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d-cq)}\right)^n.
\end{aligned}
\tag{2.33}
$$

Therefore we have that

$$
\begin{aligned}
&\frac{2\pi}{c} i^k \sum_{m=0}^{\infty} B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)(\tau+\frac{d}{c})} \\
&= \frac{\pi i^k}{c} B_{c,\nu,0,j,l} + \sideset{}{^*}\sum_{q=-\infty}^{\infty} e^{2\pi i q m_j} (c\tau+d-cq)^k \sum_{n=k+1}^{\infty} \frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d-cq)}\right)^n,
\end{aligned}
\tag{2.34}
$$

and we are done. (2.25) and Lemma 2.4 imply that

$$
\begin{aligned}
\mathcal{R}_\nu^{(j,l)}(\tau) = \mathcal{K}_\nu^{(j,l)} + \sum_{c=1}^{\infty}\sum_{d\in D_c} v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}} \sideset{}{^*}\sum_{q=-\infty}^{\infty} e^{2\pi i m_j q} (c\tau+d-cq)^k \times \\
\sum_{n=k+1}^{\infty} \frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d-cq)}\right)^n,
\end{aligned}
\tag{2.35}
$$

where

$$\mathcal{K}_\nu^{(j,l)} = \begin{cases} \sum_{c=1}^{\infty} \frac{\pi i^k}{c} A_{c,\nu,0,j,l} B_{c,\nu,0,j,l}, & m_j = 0 \\ 0, & \text{otherwise} \end{cases}, \tag{2.36}$$

and

$$D_c = \left\{ d| \quad \exists V_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1), 0 \le -d < c, 0 \le a < c \right\}. \tag{2.37}$$

Now let $d' = d - cq$ and

$$V_{c,d'} = V_{c,d} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & * \\ c & d' \end{pmatrix}. \tag{2.38}$$

As q runs through all integers and as d runs through the set D_c, d' assumes exactly once each value in D^c, where

$$D^c = \left\{ d'| \quad \exists V_{c,d'} = \begin{pmatrix} a & * \\ c & d' \end{pmatrix} \in \Gamma(1), 0 \le a < c \right\}. \tag{2.39}$$

Now by (2.38) we have that

$$\begin{aligned} &v^{-1}(V_{c,d})\rho^{-1}(V_{c,d}) \\ &= v^{-1}\left(V_{c,d'}\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}\right)\rho^{-1}\left(V_{c,d'}\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}\right) \\ &= v^{-1}(V_{c,d'})\begin{pmatrix} e^{-2\pi i m_1 q} & & \\ & \ddots & \\ & & e^{-2\pi i m_p q} \end{pmatrix}\rho^{-1}(V_{c,d'}) \\ &= v^{-1}(V_{c,d'})\begin{pmatrix} e^{-2\pi i m_1 q} & & \\ & \ddots & \\ & & e^{-2\pi i m_p q} \end{pmatrix}\begin{pmatrix} x_{c,d'}^{(1,1)} & \cdots & x_{c,d'}^{(1,p)} \\ \vdots & \ddots & \vdots \\ x_{c,d'}^{(p,1)} & \cdots & x_{c,d'}^{(p,p)} \end{pmatrix}. \end{aligned}$$

Therefore

$$v^{-1}(V_{c,d})x_{c,d}^{(j,l)} = v^{-1}(V_{c,d'})e^{-2\pi i m_j q}x_{c,d'}^{(j,l)}. \tag{2.40}$$

Now put

$$\begin{aligned} \mathcal{W}_\nu^{(j,l)} &= \delta_{jl}e^{2\pi i(m_j+\nu)\tau} + \mathcal{R}_\nu^{(j,l)} - \mathcal{K}_\nu^{(j,l)} \\ &= \mathcal{T}_\nu^{(j,l)}(\tau) - \mathcal{K}_\nu^{(j,l)}, \end{aligned} \tag{2.41}$$

and write $\mathcal{W}_\nu^{(j,l)}(\tau)$ using (2.35) (with d' replaced by d):

$$\mathcal{W}_\nu^{(j,l)}(\tau) - \delta_{jl} e^{2\pi i (m_j+\nu)\tau} = \sum_{c=1}^{\infty} \lim_{N\to\infty} \sum_{\substack{d\,=\,-N \\ d\,\in\, D^c}}^{N} v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}} (c\tau+d)^k \times \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau+d)} \right)^n . \tag{2.42}$$

To continue we will show that the series

$$\sum_{c=1}^{\infty} \lim_{N\to\infty} \sum_{\substack{d\,=\,-N \\ d\,\in\, D^c}}^{N} \frac{v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}}}{c^{k+1}(ciy+d)} \tag{2.43}$$

converges. To do so we write (2.43) as

$$\sum_{c=1}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{d\in D_c} \frac{v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}} e^{2\pi i m_j q}}{c^{k+2}\left(iy + \frac{d}{c} - q\right)}, \tag{2.44}$$

and apply the Lipschitz summation formula (2.27), for $m_j > 0$, $n = 0$ and $\tau = iy + \frac{d}{c}$ to get

$$\sum_{c=1}^{\infty} \sum_{d\in D_c} \frac{-2\pi i}{c^{k+2}} v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}} \sum_{m=0}^{\infty} e^{2\pi i (m+mj)(iy+\frac{d}{c})}. \tag{2.45}$$

For $m_j = 0$, we get a similar result. Now we see easily that (2.43) converges by applying a modified version of Lemma 2.5 in [6]:

Lemma 2.5 *If $k > 2\delta > 0$, then the sum*

$$\sum_{c=1}^{\infty} \left| \sum_{d\in D_c} v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}} \right| c^{-k-2} \tag{2.46}$$

converges.

Proof 2.5

In the proof of Lemma 2.5 in [6], Knopp shows that for $k > 0$

$$\sum_{c=1}^{\infty} \sum_{d\in D_c} c^{-k-2} \tag{2.47}$$

converges. This, combined with the fact that $k > 2\delta$ and (1.42), proves the lemma.

Now we will show that the following series

$$\sum_{n=k+2}^{\infty}\sum_{c=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{d\,=\,-N\\ d\,\in\,D^c}}^{N} v^{-1}(V_{c,d})x_{c,d}^{(j,l)}e^{2\pi i a\frac{m_l+\nu}{c}}\,(c\tau+d)^k\,\frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d)}\right)^n \tag{2.48}$$

is an absolutely convergent triple sum. To do so we rewrite (2.48) as

$$\sum_{n=k+2}^{\infty}\frac{1}{n!}\,(2\pi i(-\nu-m_l))^n\sum_{c=1}^{\infty}\sum_{d\in D_c}\frac{v^{-1}(V_{c,d})x_{c,d}^{(j,l)}e^{2\pi i a\frac{m_l+\nu}{c}}}{c^{2n-k}}\sum_{q=-\infty}^{\infty}\frac{e^{2\pi i m_j q}}{(\tau+\frac{d}{c}-q)^{n-k}}. \tag{2.49}$$

Now applying Lemma 2.5 we see that (2.48) is an absolutely convergent triple sum, and since (2.43) converges we can rewrite (2.42) as

$$\begin{aligned}\mathcal{W}_\nu^{(j,l)}(\tau)-\delta_{jl}e^{2\pi i(m_j+\nu)\tau} &= \frac{(2\pi i(-\nu-m_l))^{k+1}}{(k+1)!}\sum_{c=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{d\,=\,-N\\ d\,\in\,D^c}}^{N}\frac{v^{-1}(V_{c,d})x_{c,d}^{(j,l)}e^{2\pi i a\frac{m_l+\nu}{c}}}{c^{k+1}(ciy+d)}\\ &\quad+\sum_{c=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{d\,=\,-N\\ d\,\in\,D^c}}^{N} v^{-1}(V_{c,d})x_{c,d}^{(j,l)}e^{2\pi i a\frac{m_l+\nu}{c}}\,(c\tau+d)^k\times\\ &\qquad\sum_{n=k+2}^{\infty}\frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(c\tau+d)}\right)^n.\end{aligned} \tag{2.50}$$

To continue we need Lemma 2.13 in [7], with some modifications to be aplicable in the vector-valued case.

Lemma 2.6 *Let $\tau = iy$, with $y > 0$, $k > 2\delta$, ν a negative integer and t a positive integer. Then*

$$\sum_{c=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{d\,\in\,D^c\\ |d|\,\le\,K}}\frac{v^{-1}(V_{c,d})x_{c,d}^{(j,l)}e^{2\pi i\frac{a(m_l+\nu)}{c}}}{c^{k+1}(c\tau+d)}=\lim_{K\to\infty}\sum_{\substack{c\,\in\,\mathbb{Z}\\ 0\,<\,c\,\le\,tK}}\sum_{\substack{d\,\in\,D^c\\ |d|\,\le\,K}}\frac{v^{-1}(V_{c,d})x_{c,d}^{(j,l)}e^{2\pi i\frac{a(m_l+\nu)}{c}}}{c^{k+1}(c\tau+d)}. \tag{2.51}$$

Proof 2.6

Following Rademacher's proof [17], we will show that

$$\lim_{K\to\infty} \sum_{\substack{c\,\in\,\mathbb{Z} \\ 0\,<\,c\,\leq\,tK}} \lim_{N\to\infty} \sum_{\substack{d\,\in\,D^c \\ K\,<\,|d|\,\leq\,N}} \frac{v^{-1}(V_{c,d})x^{(j,l)}_{c,d}e^{2\pi i\frac{a(m_l+\nu)}{c}}}{c^{k+1}(c\tau+d)} = 0. \tag{2.52}$$

Note that

$$v^{-1}(V_{c,d-cq})x^{(j,l)}_{c,d-cq} = v^{-1}(V_{c,d})x^{(j,l)}_{c,d}e^{2\pi iqm_l}. \tag{2.53}$$

First we will show the result for $m_l > 0$. Replace d by $d-cq$ and rewrite the inner sum as

$$\frac{1}{c^{k+1}} \sum_{\substack{d\,\in\,D_c \\ K\,<\,|d|\,\leq\,N}} v^{-1}(V_{c,d})x^{(j,l)}_{c,d}e^{2\pi i\frac{a(m_l+\nu)}{c}}\left[S_1+S_2\right], \tag{2.54}$$

where

$$S_1 = \sum_{-\infty\leq q<\frac{-K+d}{c}} \frac{e^{2\pi iqm_l}}{c\tau+d-cq} \tag{2.55}$$

and

$$S_2 = \sum_{\frac{K+d}{c}<q\leq\infty} \frac{e^{2\pi iqm_l}}{c\tau+d-cq}. \tag{2.56}$$

Then,

$$|S_1|+|S_2| < \frac{2(t+1)t^{1-s}}{|\sin\pi m_l|c^{1-s}}K^{-s}, \tag{2.57}$$

where

$$s = \min\left\{1, \frac{k-2\delta}{2}\right\}. \tag{2.58}$$

Therefore we have that

$$\begin{aligned} &\left|\lim_{N\to\infty} \sum_{\substack{d\,\in\,D^c \\ K\,<\,|d|\,\leq\,N}} \frac{v^{-1}(V_{c,d})x^{(j,l)}_{c,d}e^{2\pi i\frac{a(m_l+\nu)}{c}}}{c^{k+1}(c\tau+d)}\right| \\ &< \frac{2(t+1)t^{1-s}}{|\sin\pi m_l|c^{2\delta+2+s}} \sum_{d\,\in\,D_c} K^{-s}|x^{(j,l)}_{c,d}| \\ &< C^*K^{-s} \sum_{d\,\in\,D_c} \frac{1}{c^{2+s}}, \end{aligned} \tag{2.59}$$

where C^* is a constant. The rest of the argument is the same as in [7].

If $m_l = 0$, write

$$\lim_{N\to\infty} \sum_{\substack{d \in D^c \\ K < |d| \le N}} \frac{v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i \frac{a(m_l+\nu)}{c}}}{c^{k+1}(c\tau+d)} = \frac{1}{c^{k+1}} \sum_{d\in D_c} v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i \frac{a(m_l+\nu)}{c}} \left[S_1' + S_2' + S_3'\right], \tag{2.60}$$

where

$$S_1' = \lim_{N\to\infty} \sum_{\frac{K-d}{c} \le |q| \le \frac{N+d}{c}} (ciy + d - cq)^{-1}, \tag{2.61}$$

$$S_2' = \lim_{N\to\infty} \sum_{\frac{-N+d}{c} \le q < \frac{-N-d}{c}} (ciy + d - cq)^{-1} \tag{2.62}$$

and

$$S_3' = \sum_{\frac{K+d}{c} < q \le \frac{K-d}{c}} (ciy + d - cq)^{-1}. \tag{2.63}$$

Now apply the same argument as in [7] using the estimate (1.42) and the fact that $k > 2\delta$.

Now using Lemma 2.6 with $t = 1$, we can rewrite (2.50) as

$$\begin{aligned} &\mathcal{W}_\nu^{(j,l)}(\tau) - \delta_{jl} e^{2\pi i (m_j+\nu)\tau} \\ &= \lim_{K\to\infty} \sum_{\substack{c \in \mathbb{Z} \\ 0 < c \le K}} \sum_{\substack{d \in D^c \\ |d| \le K}} \frac{v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}} \left(2\pi i(-\nu - m_l)\right)^{k+1}}{c^{k+1}(c\tau+d)(k+1)!} \\ &+ \lim_{K\to\infty} \sum_{\substack{c \in \mathbb{Z} \\ 0 < c \le K}} \sum_{\substack{d \in D^c \\ |d| \le K}} v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}} (c\tau+d)^k \times \\ &\qquad \sum_{n=k+2}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau+d)}\right)^n . \end{aligned} \tag{2.64}$$

This is

$$\mathcal{W}_\nu^{(j,l)}(\tau) - \delta_{jl} e^{2\pi i (m_j+\nu)\tau} = \lim_{K\to\infty} \sum_{\substack{c \in \mathbb{Z} \\ 0 < c \le K}} \sum_{\substack{d \in D^c \\ |d| \le K}} v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} e^{2\pi i a \frac{m_l+\nu}{c}} (c\tau+d)^k \times \left(e^{\frac{2\pi i(-\nu - m_l)}{c(c\tau+d)}} - \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(c\tau+d)}\right)^n \right). \tag{2.65}$$

Put

$$\mathcal{S}^{(j,l)}_{\nu,K}(\tau) = \delta_{jl} e^{2\pi i (m_j+\nu)\tau} + \sum_{\substack{c\,\in\,\mathbb{Z}\\ 0\,<\,c\,\le\,K}} \sum_{\substack{d\,\in\,D^c\\ |d|\,\le\,K}} v^{-1}(V_{c,d}) x^{(j,l)}_{c,d} e^{2\pi i a \frac{m_l+\nu}{c}} (c\tau+d)^k e^{\frac{2\pi i(-\nu-m_l)}{c(c\tau+d)}}. \tag{2.66}$$

Since $V_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, we have that

$$-\frac{a}{c} + \frac{1}{c\,(c\tau+d)} = \frac{-ac\tau - ad + ad - bc}{c\,(c\tau+d)} = -V_{c,d}\tau, \tag{2.67}$$

and therefore

$$e^{2\pi i a\frac{m_l+\nu}{c}} e^{\frac{2\pi i(-\nu-m_l)}{c(c\tau+d)}} = e^{2\pi i(\nu+m_l)V_{c,d}\tau}. \tag{2.68}$$

Thus

$$\mathcal{S}^{(j,l)}_{\nu,K}(\tau) = \delta_{jl} e^{2\pi i (m_l+\nu)\tau} + \sum_{\substack{c\,\in\,\mathbb{Z}\\ 0\,<\,c\,\le\,K}} \sum_{\substack{d\,\in\,D^c\\ |d|\,\le\,K}} v^{-1}(V_{c,d}) x^{(j,l)}_{c,d} (c\tau+d)^k e^{2\pi i(\nu+m_l)V_{c,d}\tau}. \tag{2.69}$$

Now we include the first term in the summation. Since $v(I)\rho(I) = I$, we have that

$$\delta_{jl} e^{2\pi i(m_l+\nu)\tau} = v^{-1}(I) x^{(j,l)}_{0,1} e^{2\pi i(m_l+\nu)I\tau}. \tag{2.70}$$

Next we include the pair $(c,d) = (0,1)$ and we get

$$\mathcal{S}^{(j,l)}_{\nu,K}(\tau) = \sum_{\substack{c\,\in\,\mathbb{Z}\\ 0\,\le\,c\,\le\,K}} \sum_{\substack{d\,\in\,D^c\\ |d|\,\le\,K\\ (c,\,d)\,\neq\,(0,\,-1)}} v^{-1}(V_{c,d}) x^{(j,l)}_{c,d} (c\tau+d)^k e^{2\pi i(\nu+m_l)V_{c,d}\tau}. \tag{2.71}$$

For every transformation $V_{c,d}$, included in (2.71), we next include the transformation $-V_{c,d} = V_{-c,-d}$. Since we have that

$$v^{-1}(V_{c,d})\rho^{-1}(V_{c,d})\,(c\tau+d)^k = v^{-1}(-V_{c,d})\rho^{-1}(-V_{c,d})\,(-c\tau-d)^k \tag{2.72}$$

and

$$e^{2\pi i(m_l+\nu)V_{c,d}\tau} = e^{2\pi i(m_l+\nu)\left(-V_{c,d}\tau\right)}, \tag{2.73}$$

we see that if we make the summation including the transformations $-V_{c,d} = V_{-c,-d}$, every term of $\mathcal{S}^{(j,l)}_{\nu,K}(\tau)$ occurs twice and therefore

$$\mathcal{S}^{(j,l)}_{\nu,K}(\tau) = \frac{1}{2} \sum_{\substack{c \in \mathbb{Z} \\ |c| \leq K}} \sum_{\substack{d \in D^c \\ |d| \leq K}} v^{-1}(V_{c,d}) x^{(j,l)}_{c,d} (c\tau + d)^k \, e^{2\pi i(\nu + m_l)V_{c,d}\tau}. \tag{2.74}$$

Now let $\mathcal{S}_{\nu,K}(\tau)$ be the matrix with components given by (2.74). Then we define the matrix $\mathcal{M}_{S,\nu,K}(\tau) = v^{-1}(S)\rho^{-1}(S)\tau^k \mathcal{S}_{\nu,K}(S\tau)$, given by

$$\begin{aligned}
&\mathcal{M}^{(j,l)}_{S,\nu,K}(\tau) \\
&\quad = \frac{1}{2} \sum_{\substack{c \in \mathbb{Z} \\ |c| \leq K}} \sum_{\substack{d \in D^c \\ |d| \leq K}} v^{-1}(S) v^{-1}(V_{c,d}) \sum_{s=1}^{p} x^{(j,s)}_S x^{(s,l)}_{c,d} \tau^k \, (cS\tau + d)^k \, e^{2\pi i(m_l + \nu)V_{c,d}S\tau} \\
&\quad = \frac{1}{2} \sum_{\substack{c \in \mathbb{Z} \\ |c| \leq K}} \sum_{\substack{d \in D^c \\ |d| \leq K}} v^{-1}(V_{c,d}S) x^{(j,l)}_{V_{c,d}S} \, (d\tau - c)^k \, e^{2\pi i(m_l + \nu)V_{c,d}S\tau}.
\end{aligned}$$

Now we make the transformation $c' = d$, and $d' = -c$. This is a 1-1 correspondence between the pairs $\{(c,d) | c \in \mathbb{Z}, d \in D^c\}$ and the pairs $\{(c', d') | c' \in \mathbb{Z}, d' \in D^{c'}\}$. Then

$$\begin{aligned}
&\mathcal{M}^{(j,l)}_{V,\nu,K}(\tau) \\
&\quad = \frac{1}{2} \sum_{\substack{c' \in \mathbb{Z} \\ |c'| \leq K}} \sum_{\substack{d' \in D^{c'} \\ |d'| \leq K}} v^{-1}(V_{c',d'}) x^{(j,l)}_{c',d'} \, (c'\tau + d')^k \, e^{2\pi i(m_l + \nu)V_{c',d'}\tau} \\
&\quad = \delta_{jl} e^{2\pi i(m_l + \nu)\tau} + \sum_{\substack{c \in \mathbb{Z} \\ 0 < c \leq K}} \sum_{\substack{d \in D^c \\ |d| \leq K}} v^{-1}(V_{c,d}) x^{(j,l)}_{c,d} \, (c\tau + d)^k \, e^{2\pi i(m_l + \nu)V_{c,d}\tau}.
\end{aligned}$$

Now put

$$\mathcal{U}_{S,\nu}(\tau) = v^{-1}(S)\rho^{-1}(S)\tau^k \mathcal{W}_\nu(-\frac{1}{\tau}), \tag{2.75}$$

where $\mathcal{T}_\nu(\tau)$ is given by (2.24), $\mathcal{W}_\nu(\tau)$ by (2.41) and $\mathcal{K}_\nu$ is the matrix given by (2.36). Then we have

$$
\begin{aligned}
&\mathcal{U}_{S,\nu}^{(j,l)}(\tau) \\
&= \lim_{K\to\infty}\Bigg\{ \mathcal{M}_{S,\nu,K}^{(j,l)}(\tau) - \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d\in D^c\\ |d|\le K}} v^{-1}(V_{c,d}S)x_{V_{c,d}S}^{(j,l)}\tau^k \\
&\qquad (cS\tau+d)^k\, e^{2\pi i\frac{a(m_l+\nu)}{c}} \sum_{n=0}^{k}\frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(cS\tau+d)}\right)^n \Bigg\} \\
&= \delta_{jl}e^{2\pi i(m_l+\nu)\tau} + \lim_{K\to\infty}\Bigg\{ \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d\in D^c\\ |d|\le K}} v^{-1}(V_{c,d})x_{c,d}^{(j,l)}\,(c\tau+d)^k\, e^{2\pi i(m_l+\nu)V_{c,d}\tau} \\
&\qquad - \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d\in D^c\\ |d|\le K}} v^{-1}(V_{c,d}S)x_{V_{c,d}S}^{(j,l)}\tau^k\,(cS\tau+d)^k \\
&\qquad e^{2\pi i\frac{a(m_l+\nu)}{c}} \sum_{n=0}^{k}\frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(cS\tau+d)}\right)^n \Bigg\}.
\end{aligned}
\tag{2.76}
$$

Also Lemma 2.6 implies

$$
\begin{aligned}
&\mathcal{W}_{\nu}^{(j,l)}(\tau) \\
&\quad = \delta_{jl}e^{2\pi i(m_l+\nu)\tau} + \lim_{K\to\infty}\Bigg\{ \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d\in D^c\\ |d|\le K}} v^{-1}(V_{c,d})x_{c,d}^{(j,l)}\,(c\tau+d)^k\, e^{2\pi i(m_l+\nu)V_{c,d}\tau} \\
&\quad - \sum_{\substack{c\in\mathbb{Z}\\ 0<c\le K}} \sum_{\substack{d\in D^c\\ |d|\le K}} v^{-1}(V_{c,d})x_{c,d}^{(j,l)}\,(c\tau+d)^k\, e^{2\pi i\frac{a(m_l+\nu)}{c}} \sum_{n=0}^{k}\frac{1}{n!}\left(\frac{2\pi i(-\nu-m_l)}{c(cS\tau+d)}\right)^n \Bigg\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathcal{W}_{\nu}^{(j,l)}(\tau) - \mathcal{U}_{S,\nu}^{(j,l)}(\tau) \\
&= \lim_{K\to\infty} \Bigg\{ \sum_{\substack{c\,\in\,\mathbb{Z} \\ 0\,<\,c\,\leq\,K}} \sum_{\substack{d\,\in\,D^c \\ |d|\,\leq\,K}} v^{-1}(V_{c,d}S) x_{V_{c,d}S}^{(j,l)} \tau^k \left(cS\tau + d\right)^k \\
&\qquad e^{2\pi i \frac{a(m_l+\nu)}{c}} \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(cS\tau + d)} \right)^n \\
&- \sum_{\substack{c\,\in\,\mathbb{Z} \\ 0\,<\,c\,\leq\,K}} \sum_{\substack{d\,\in\,D^c \\ |d|\,\leq\,K}} v^{-1}(V_{c,d}) x_{c,d}^{(j,l)} \left(c\tau + d\right)^k e^{2\pi i \frac{a(m_l+\nu)}{c}} \sum_{n=0}^{k} \frac{1}{n!} \left(\frac{2\pi i(-\nu - m_l)}{c(cS\tau + d)} \right)^n \Bigg\}.
\end{aligned}
\tag{2.77}
$$

Since the factor τ^k combines with $(cS\tau + d)^{k-n}$, it produces a polynomial of degree at most k. On the other hand, the limit of a sequence of polynomials of degree at most k converging at $k+1$ points is a polynomial of degree at most k. Now put

$$
\mathcal{Y}_{S,\nu}(\tau) = \mathcal{K}_{\nu} - v^{-1}(S)\tau^k \rho^{-1}(S)\mathcal{K}_{\nu}. \tag{2.78}
$$

Clearly $\mathcal{Y}_{S,\nu}^{(j,l)}(\tau)$ is a polynomial in τ of degree at most k, and so is

$$
\begin{aligned}
\mathcal{Q}_{S,\nu}^{(j,l)}(\tau) &= \mathcal{W}_{\nu}^{(j,l)}(\tau) - \mathcal{U}_{S,\nu}^{(j,l)}(\tau) + \mathcal{Y}_{S,\nu}^{(j,l)}(\tau) \\
&= \mathcal{T}_{\nu}^{(j,l)}(\tau) - v^{-1}(S)\tau^k \sum_{s=1}^{p} x_S^{(j,s)} \mathcal{T}_{\nu}^{(s,l)}(S\tau).
\end{aligned}
\tag{2.79}
$$

By (2.24) we see that

$$
\begin{aligned}
&F(\tau) - v^{-1}(S)\tau^k \rho^{-1}(S) F(S\tau) \\
&= \sum_{\mu\leq\nu<0} \left(\mathcal{W}_{\nu}(\tau) - \mathcal{U}_{S,\nu}(\tau)\right) \begin{pmatrix} b_{\nu}^{(1)} \\ \vdots \\ b_{\nu}^{(p)} \end{pmatrix} + \sum_{\mu\leq\nu<0} \mathcal{Y}_{S,\nu}(\tau) \begin{pmatrix} b_{\nu}^{(1)} \\ \vdots \\ b_{\nu}^{(p)} \end{pmatrix}
\end{aligned}
\tag{2.80}
$$

is a column vector of polynomials of degree at most k. Now, for all $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$, put

$$
\mathcal{U}_{V,\nu}(\tau) = v^{-1}(V)\rho^{-1}(V)\left(\gamma\tau + \delta\right)^k \mathcal{W}_{\nu}(\tau), \tag{2.81}
$$

$$\mathcal{Y}_{V,\nu}(\tau) = \mathcal{K}_\nu - \upsilon^{-1}(V)(\gamma\tau+\delta)^k \rho^{-1}(V)\mathcal{K}_\nu \tag{2.82}$$

and

$$\begin{aligned} \mathcal{Q}_{V,\nu}^{(j,l)}(\tau) &= \mathcal{W}_\nu^{(j,l)}(\tau) - \mathcal{U}_{V,\nu}^{(j,l)}(\tau) + \mathcal{Y}_{V,\nu}^{(j,l)}(\tau) \\ &= \mathcal{T}_\nu^{(j,l)}(\tau) - \upsilon^{-1}(V)\,(\gamma\tau+\delta)^k \sum_{s=1}^{p} x_V^{(j,s)} \mathcal{T}_\nu^{(s,l)}(V\tau). \end{aligned} \tag{2.83}$$

We want to show that the matrix $\mathcal{Q}_{V,\nu}(\tau) = \mathcal{T}_\nu(\tau) - \mathcal{T}_\nu \mid_{-k,\upsilon,\rho} V(\tau)$, given by (2.83), is a matrix of polynomials of degree at most k. We will prove this by induction on $L(V)$, the length of V when written in terms of S and T. By (2.79), we know it is true for S. By (2.24) and the normalization (1.6) it is clear that $\mathcal{T}_\nu(\tau) = \mathcal{T}_\nu \mid_{-k,\upsilon,\rho} T(\tau)$. Now, assume for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ that $\mathcal{Q}_{M,\nu}(\tau)$ is a matrix of polynomials of degree at most k, we want to show that:

1. $\mathcal{T}_\nu(\tau) - \upsilon^{-1}(MS)\rho^{-1}(MS)\,(d\tau - c)^k\, \mathcal{T}_\nu(MS\tau) = \mathcal{Q}_{MS,\nu}(\tau)$, where $\mathcal{Q}_{MS,\nu}(\tau)$ is a matrix of polynomials of degree at most k, and

2. $\mathcal{T}_\nu(\tau) - \upsilon^{-1}(MT)\rho^{-1}(MT)\,(c\tau + d + c)^k\, \mathcal{T}_\nu(MT\tau) = \mathcal{Q}_{MT,\nu}(\tau)$, where $\mathcal{Q}_{MT,\nu}(\tau)$ is a matrix of polynomials of degree at most k.

1. Since $\mathcal{Q}_{M,\nu}(\tau)$ and $\mathcal{Q}_{S,\nu}(\tau)$ are matrices of polynomials of degree $\leq k$, we have that

$$\begin{aligned} &\mathcal{T}_\nu(\tau) - \upsilon^{-1}(MS)\rho^{-1}(MS)\,(d\tau - c)^k\, \mathcal{T}_\nu(MS\tau) \\ &= \mathcal{T}_\nu(\tau) - \tau^k \upsilon^{-1}(S)\rho^{-1}(S)\left((cS\tau + d)^k\, \upsilon^{-1}(M)\rho^{-1}(M)\mathcal{T}_\nu(M(S\tau))\right) \\ &= \mathcal{T}_\nu(\tau) - \tau^k \upsilon^{-1}(S)\rho^{-1}(S)\left(\mathcal{T}_\nu(S\tau) - \mathcal{Q}_{M,\nu}(S\tau)\right) \\ &= \mathcal{Q}_{S,\nu}(\tau) + \tau^k \upsilon^{-1}(S)\rho^{-1}(S)\mathcal{Q}_{M,\nu}(S\tau). \end{aligned} \tag{2.84}$$

The first term is a matrix of polynomials. In the second term τ^k combines with $\left(-\frac{1}{\tau}\right)^{k-n}$, for $0 \leq n \leq k$, forming a matrix of polynomials.

2. Similarly, using the fact that $\mathcal{Q}_{T,\nu}(\tau) \equiv 0$, we have

$$\begin{aligned}
&\mathcal{T}_\nu(\tau) - v^{-1}(MT)\rho^{-1}(MT)\,(c\tau+d)^k\,\mathcal{T}_\nu(MT\tau) \\
&= \mathcal{T}_\nu(\tau) - v^{-1}(T)\rho^{-1}(T)\left((cT\tau+d)^k\,v^{-1}(M)\rho^{-1}(M)\mathcal{T}_\nu(M(T\tau))\right) \\
&= \mathcal{T}_\nu(\tau) - v^{-1}(T)\rho^{-1}(T)\,(\mathcal{T}_\nu(T\tau) - \mathcal{Q}_{M,\nu}(T\tau)) \\
&= v^{-1}(T)\rho^{-1}(T)\mathcal{Q}_{M,\nu}(T\tau),
\end{aligned} \tag{2.85}$$

which is a matrix of polynomials of degree at most k, sice $\mathcal{Q}_{M,\nu}(\tau)$ is.

Therefore we have shown that if $\mathcal{Q}_{M,\nu}(\tau)$ is matrix of polynomials in τ of degree at most k, so are $\mathcal{Q}_{MS,\nu}(\tau)$ and $\mathcal{Q}_{MT,\nu}(\tau)$. We have also shown that $\mathcal{Q}_{S,\nu}(\tau)$ is a matrix of polynomials in τ of degree at most k, and that $\mathcal{Q}_{T,\nu}(\tau) \equiv 0$. It is clear by (2.85) that $\mathcal{Q}_{T^n,\nu}(\tau) \equiv 0$, and therefore if $L(V) = 1$, we have that either $V = S$ or $V = T^n$, in either case we have that $\mathcal{Q}_{V,\nu}(\tau)$ is matrix of polynomials in τ of degree at most k. Assume that for $L(V) = r$, we have that $\mathcal{Q}_{V,\nu}(\tau)$ is matrix of polynomials in τ of degree at most k. Then since $\mathcal{Q}_{VS,\nu}(\tau)$ and $\mathcal{Q}_{VT^n,\nu}(\tau)$ are also matrices of polynomials in τ of degree at most k, we have that the result is also true for $L(V) = r+1$. Therefore for all $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$, we have that

$$\mathcal{Q}_{V,\nu}(\tau) = \mathcal{T}_\nu(\tau) - \mathcal{T}_\nu \mid_{-k,v,\rho} V(\tau), \tag{2.86}$$

which is a polynomial of degree at most k. Thus by (2.83) and (2.23), we have

$$\begin{aligned}
&F(\tau) - v^{-1}(V)(\gamma\tau+\delta)^k\rho^{-1}(V)F(V\tau) \\
&= \sum_{\mu\le\nu<0} (\mathcal{W}_\nu(\tau) - \mathcal{U}_{V,\nu}(\tau)) \begin{pmatrix} b_\nu^{(1)} \\ \vdots \\ b_\nu^{(p)} \end{pmatrix} + \sum_{\mu\le\nu<0} \mathcal{Y}_{V,\nu}(\tau) \begin{pmatrix} b_\nu^{(1)} \\ \vdots \\ b_\nu^{(p)} \end{pmatrix} \\
&= \sum_{\mu\le\nu<0} \mathcal{Q}_{M,\nu}(\tau) \begin{pmatrix} b_\nu^{(1)} \\ \vdots \\ b_\nu^{(p)} \end{pmatrix} \\
&= Q_M(\tau, k, v, \rho),
\end{aligned} \tag{2.87}$$

which is a vector of polynomials of degree at most k, since the right hand side is a linear combination of polynomials of degree at most k. The proof of Theorem 2.1 is complete.

2.2 Construction of a vector-valued modular form of negative weight $-k$

So far we have that

$$F(\tau)-v^{-1}(M)\rho^{-1}(M)\left(\gamma\tau+\delta\right)^{k}F(M\tau)=Q_M(\tau,k,v,\rho),\quad M=\begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}\in\Gamma(1), \tag{2.88}$$

where $Q_M(\tau,k,v,\rho)$ is a column vector of polynomials of degree at most k. We ask whether we can construct a vector-valued modular form of negative weight $-k$ by choosing appropriately the coefficients of the principal part in (2.1). It turns out that if there are enough coefficients in the principal part, this is in fact possible. Note that the number needed depends on k.

Let $T=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By (2.22) , it is easy to see that for $n\in\mathbb{Z}$ we have

$$F(T^n\tau)=\begin{pmatrix} e^{2\pi i n m_1} & & \\ & \ddots & \\ & & e^{2\pi i n m_p} \end{pmatrix}F(\tau)=v(T^n)\rho(T^n)F(\tau). \tag{2.89}$$

This implies that $Q_{T^n}(\tau,k,v,\rho)\equiv 0$.

Since all the elements of $\Gamma(1)$ can be written as a product of T^n and S for $n\in\mathbb{Z}$, we want to find $b_{-1}^{(j)},\ldots,b_{\mu}^{(j)}$ such that

$$v^{-1}(S)\rho^{-1}(S)\tau^{k}F(S\tau)-F(\tau)=0, \tag{2.90}$$

or what is the same

$$v^{-1}(S)\rho^{-1}(S)\tau^k F(S\tau) - F(\tau) = \sum_{\mu\le\nu<0} \mathcal{Q}_{S,\nu}(\tau)\begin{pmatrix} b_\nu^{(1)} \\ \vdots \\ b_\nu^{(p)}\end{pmatrix}$$

$$= \begin{pmatrix} Q_S^{(1)}(\tau,k,v,\rho) \\ \vdots \\ Q_S^{(p)}(\tau,k,v,\rho)\end{pmatrix} \tag{2.91}$$

$$\equiv \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix}.$$

If we do so, by the proof that the right hand side of (2.83) is a polynomial of degree at most k, we will have a function $F(\tau)$ with the following transformation law, for all $M \in \Gamma(1)$,

$$F(\tau) = v^{-1}(M)\rho^{-1}(M)\left(\gamma\tau+\delta\right)^k F(M\tau). \tag{2.92}$$

Also since $F(\tau)$ is regular in $\mathcal{H}$, and has the Fourier expansion at ∞ given by (2.1):

$$F^{(j)}(\tau) = \sum_{\mu\le\nu<0} b_\nu^{(j)} e^{2\pi i(m_j+\nu)\tau} + \sum_{m=0}^{\infty}\sum_{\mu\le\nu<0}\sum_{l=1}^{p} b_\nu^{(l)} 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} A_{c,\nu,m,j,l}\, B_{c,\nu,m,j,l} e^{2\pi i(m+m_j)\tau}, \tag{2.93}$$

it follows that $F(\tau)$ is a vector-valued modular form of weight $-k$.

Now if we replace τ by $S\tau$ in (2.91), we see that

$$v^{-1}(S)\rho^{-1}(S)\tau^k \begin{pmatrix} Q_S^{(1)}(\frac{-1}{\tau},k,v,\rho) \\ \vdots \\ Q_S^{(p)}(\frac{-1}{\tau},k,v,\rho)\end{pmatrix} = \begin{pmatrix} -Q_S^{(1)}(\tau,k,v,\rho) \\ \vdots \\ -Q_S^{(p)}(\tau,k,v,\rho)\end{pmatrix}, \tag{2.94}$$

or, what is the same,

$$v^{-1}(S)\rho^{-1}(S)\tau^k \sum_{s=1}^{r} \mathcal{Q}_{S,-s}\left(\frac{-1}{\tau}\right)\begin{pmatrix} b_{-s}^{(1)} \\ \vdots \\ b_{-s}^{(p)}\end{pmatrix} = -\sum_{s=1}^{r}\mathcal{Q}_{S,-s}(\tau)\begin{pmatrix} b_{-s}^{(1)} \\ \vdots \\ b_{-s}^{(p)}\end{pmatrix}. \tag{2.95}$$

By (2.94), we see that the zeros of $Q_S^{(j)}(\tau,k,v,\rho)$ occur in pairs, except for $\tau=\pm i$, since $det(\rho^{-1}(S))\neq 0$.

Now, since

$$\begin{pmatrix} \mathcal{Q}_{S,-1}(\tau_1) & \mathcal{Q}_{S,-2}(\tau_1) & \dots & \mathcal{Q}_{S,-r}(\tau_1) \\ \mathcal{Q}_{S,-1}(\tau_2) & \mathcal{Q}_{S,-2}(\tau_2) & \dots & \mathcal{Q}_{S,-r}(\tau_2) \\ \vdots & & \ddots & \vdots \\ \mathcal{Q}_{S,-1}(\tau_n) & \mathcal{Q}_{S,-2}(\tau_n) & \dots & \mathcal{Q}_{S,-r}(\tau_n) \end{pmatrix} \begin{pmatrix} b_{-1}^{(1)} \\ \vdots \\ b_{-1}^{(p)} \\ b_{-2}^{(1)} \\ \vdots \\ b_{-2}^{(p)} \\ \vdots \\ b_{-r}^{(p)} \end{pmatrix} = \begin{pmatrix} Q_S^{(1)}(\tau_1,k,v,\rho) \\ \vdots \\ Q_S^{(p)}(\tau_1,k,v,\rho) \\ Q_S^{(1)}(\tau_2,k,v,\rho) \\ \vdots \\ Q_S^{(p)}(\tau_2,k,v,\rho) \\ \vdots \\ Q_S^{(p)}(\tau_n,k,v,\rho) \end{pmatrix}, \tag{2.96}$$

we set up the following system of equations:

$$\begin{pmatrix} \mathcal{Q}_{S,-1}(\tau_1) & \mathcal{Q}_{S,-2}(\tau_1) & \dots & \mathcal{Q}_{S,-r}(\tau_1) \\ \mathcal{Q}_{S,-1}(\tau_2) & \mathcal{Q}_{S,-2}(\tau_2) & \dots & \mathcal{Q}_{S,-r}(\tau_2) \\ \vdots & & \ddots & \vdots \\ \mathcal{Q}_{S,1}(\tau_n) & \mathcal{Q}_{S,-2}(\tau_n) & \dots & \mathcal{Q}_{S,-r}(\tau_n) \end{pmatrix} \begin{pmatrix} b_{-1}^{(1)} \\ \vdots \\ b_{-1}^{(p)} \\ b_{-2}^{(1)} \\ \vdots \\ b_{-2}^{(p)} \\ \vdots \\ b_{-r}^{(p)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.97}$$

It is clear that if $n>k$ and $r>n$, then this system is a homogeneous system with a nontrivial solution since the number of unknowns, pr, is bigger than the number of equations, pn. Also if $n>k$, we have that all polynomials $Q_S^{(j)}(\tau,k,v,\rho)$ have $n>k$ roots and therefore they are identically zero. However we can use the fact that the zeros of $Q_S^{(j)}(\tau,k,v,\rho)$ occur in pairs, so that if τ_s is a root of $Q_S^{(j)}(\tau,k,v,\rho)$, so is $-\frac{1}{\tau_s}$. So let $\mathcal{S}$ be a set of distinct points, $\mathcal{S}=\{\tau_1,\dots,\tau_n\}$, such that if $\tau_s\in\mathcal{S}$, then $-\frac{1}{\tau_s}$ is not in $\mathcal{S}$. Now we set up the linear equation (2.97) using exclusively elements in $\mathcal{S}$. Then if $n=\left[\frac{k}{2}\right]+1$, we will have that all polynomials $Q_S^{(j)}(\tau, k, v, \rho)$ have more than k zeros, and therefore are identically zero.

We can now state the following theorem:

Theorem 2.7 *Let r be an integer greater than $[k/2]+1$. If we define $F(\tau)$ as in (2.1) with $k > 2\delta$ and $b_{-1}, \ldots, b_{-r}$ column vectors of length p satisfying (2.97), then $F(\tau)$ is a vector-valued modular form of weight $-k$.*

2.3 The supplementary series

Let m'_s and ν' be defined by

$$\begin{aligned} m'_j &= 1 - m_j, \quad \nu' = -1-\nu, \quad \text{if } m_j > 0 \\ m'_j &= -m_j, \qquad \nu' = -\nu, \qquad \text{if } m_j = 0. \end{aligned} \tag{2.98}$$

Note that $m'_j + \nu' = -(m_j + \nu)$. Further we can define

$$v'(V) = \overline{v(V)} \quad \text{and} \quad \rho'(V) = \overline{\rho(V)}. \tag{2.99}$$

Since k is an integer and v is a multiplier system in weight $-k$ for $\Gamma(1)$, it follows that v' is also a multiplier system for $\Gamma(1)$ in weight $-k$. On the other hand, since ρ is a representation for $\Gamma(1)$, we have that ρ' is also a representation. Note that

$$v'\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\rho'\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} e^{2\pi i m'_1} & & \\ & \ddots & \\ & & e^{2\pi i m'_p} \end{pmatrix}. \tag{2.100}$$

Now we define the series supplementary to $\mathcal{R}_\nu^{(j,l)}(\tau)$ as

$$\widehat{\mathcal{R}}_\nu^{(j,l)}(\tau) = \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^k}{c} \widehat{A}_{c,\nu,m,j,l}\ \widehat{B}_{c,\nu,m,j,l} e^{2\pi i (m+m'_j)\tau}, \tag{2.101}$$

where

$$\widehat{A}_{c,\nu,m,j,l} = \sum_{\substack{c,\,d \\ 0 \le -d < c \\ (c,d)=1}} \widehat{\Omega}_{c,d,l,j} e^{2\pi i \frac{dm + a\nu'}{c}}, \tag{2.102}$$

$$\widehat{B}_{c,\nu,m,j,l} = \begin{cases} \left(\frac{-\nu'-m'_l}{m+m'_j}\right)^{\frac{k+1}{2}} I_{k+1}\left(\frac{4\pi}{c}(-\nu'-m'_l)^{\frac{1}{2}}(m+m'_j)^{\frac{1}{2}}\right), & m+m'_j>0 \\ \frac{1}{(k+1)!}\left(\frac{2\pi(-\nu'-m'_l)}{c}\right)^{k+1}, & m=m'_j=0 \end{cases} \tag{2.103}$$

and

$$\widehat{\Omega}_{c,d,l,j} = \overline{\Omega}_{c,d,l,j}. \tag{2.104}$$

Note that $\widehat{\mathcal{R}}_\nu^{(j,l)}(\tau)$ is given by (2.21),and $\Omega_{c,d,l,j}$ by (1.96) but with v', ρ', ν' and m'_s replacing v, ρ, ν and m_s respectively. Similarly we can define the supplementary series $\widehat{\mathcal{T}}_\nu^{(j,l)}(\tau)$ (2.24), $\widehat{\mathcal{K}}_\nu^{(j,l)}$ (2.36), $\widehat{\mathcal{W}}_\nu^{(j,l)}(\tau)$ (2.41), $\widehat{\mathcal{Y}}_{V,\nu}^{(j,l)}(\tau)$ (2.82) and $\widehat{\mathcal{U}}_{V,\nu}^{(j,l)}(\tau)$ (2.81).

Using exactly the same arguments as before, we see that:

1. $\widehat{\mathcal{T}}_\nu^{(j,l)}(\tau)$ is regular for $\tau \in \mathcal{H}$.

2. For every $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$ we have

$$\widehat{\mathcal{T}}_\nu(\tau) - v'^{-1}(M)\rho'^{-1}(M)\,(\gamma\tau+\delta)^k\,\widehat{\mathcal{T}}_\nu(M\tau) = \widehat{\mathcal{Q}}_{M,\nu}(\tau). \tag{2.105}$$

3. The polynomials $\widehat{\mathcal{Q}}_{M,\nu}^{(j,l)}(\tau)$ are given by formulas analogous to those in (2.83):

$$\begin{aligned} \widehat{\mathcal{Q}}_{M,\nu}^{(j,l)}(\tau) &= \widehat{\mathcal{W}}_\nu^{(j,l)}(\tau) - \widehat{\mathcal{U}}_{M,\nu}^{(j,l)}(\tau) + \widehat{\mathcal{Y}}_{M,\nu}^{(j,l)}(\tau) \\ &= \widehat{\mathcal{T}}_\nu^{(j,l)}(\tau) - v'^{-1}(M)\,(\gamma\tau+\delta)^k \sum_{s=1}^{p} \overline{x_V}^{(j,s)}\widehat{\mathcal{T}}_\nu^{(s,l)}(M\tau). \end{aligned} \tag{2.106}$$

Here we have some facts:

1. From (2.42) we see that $\widehat{\mathcal{W}}_\nu^{(j,l)}(\tau) = \overline{\mathcal{W}_\nu^{(j,l)}(\overline{\tau})}$.

2. From (2.81) we see that $\widehat{\mathcal{U}}_{V,\nu}^{(j,l)}(\tau) = \overline{\mathcal{U}_{V,\nu}^{(j,l)}(\overline{\tau})}$.

3. From (2.102) we see that $\widehat{A}_{c,\nu,0,j,l} = \overline{A_{c,\nu,0,j,l}}$.

4. From (1.108) we see that $\widehat{B}_{c,\nu,0,j,l} = (-1)^{k+1}\overline{B_{c,\nu,0,j,l}}$.

5. From (2.36) and the above results we see that $\widehat{\mathcal{K}}_{\nu}^{(j,l)} = -\overline{\mathcal{K}_{\nu}^{(j,l)}}$.

6. From (2.82) and the above results we see that $\widehat{\mathcal{Y}}_{V,\nu}^{(j,l)}(\tau) = -\overline{\mathcal{Y}_{V,\nu}^{(j,l)}(\overline{\tau})}$.

7. From (2.106) and the above results we see that

$$\widehat{\mathcal{Q}}_{M,\nu}^{(j,l)}(\tau) = \overline{\mathcal{Q}_{M,\nu}^{(j,l)}(\overline{\tau})} - 2\overline{\mathcal{Y}_{M,\nu}^{(j,l)}(\overline{\tau})}. \tag{2.107}$$

Now let $b_{-1}, \ldots, b_{\mu}$ be a set of column vectors such that $b_{\nu} \in \mathbb{C}^p$, $b_{\mu}^{(j)} \neq 0$, for some $1 \leq j \leq p$, $\rho : \Gamma \longrightarrow GL(p, \mathbb{C})$ a p-dimensional complex representation, v a multiplier system, and $k > 2\delta$, $k, -\mu \in \mathbb{Z}^+$. Let $F(\tau)$ be defined as in Theorem 2.1. Then by (2.23), we have that

$$F(\tau) = \sum_{\mu \leq \nu < 0} \mathcal{T}_{\nu}(\tau) \begin{pmatrix} b_{\nu}^{(1)} \\ \vdots \\ b_{\nu}^{(p)} \end{pmatrix}. \tag{2.108}$$

We define the series $\widehat{F}(\tau)$ supplementary to $F(\tau)$ by

$$\widehat{F}(\tau) = \sum_{\mu \leq \nu < 0} \widehat{\mathcal{T}}_{\nu}(\tau) \begin{pmatrix} \overline{b_{\nu}}^{(1)} \\ \vdots \\ \overline{b_{\nu}}^{(p)} \end{pmatrix}. \tag{2.109}$$

For $M \in \Gamma(1)$ we have that

$$F(\tau) - v^{-1}(M)\rho^{-1}(M)\left(\gamma\tau + \delta\right)^{k} F(M\tau) = Q_M(\tau, k, v, \rho), \tag{2.110}$$

where by (2.87),

$$Q_M(\tau, k, v, \rho) = \sum_{\mu \leq \nu < 0} \mathcal{Q}_{M,\nu}(\tau) \begin{pmatrix} b_{\nu}^{(1)} \\ \vdots \\ b_{\nu}^{(p)} \end{pmatrix}. \tag{2.111}$$

Similarly,

$$\begin{aligned}
&\widehat{F}(\tau) - v'^{-1}(M)\rho'^{-1}(M)\,(\gamma\tau+\delta)^k\,\widehat{F}(M\tau) \\
&= \sum_{\mu\le\nu<0}\left(\widehat{\mathcal{W}}_\nu(\tau) - \widehat{\mathcal{U}}_{M,\nu}(\tau) + \widehat{\mathcal{Y}}_{M,\nu}(\tau)\right)\begin{pmatrix}\overline{b_\nu}^{(1)}\\ \vdots \\ \overline{b_\nu}^{(p)}\end{pmatrix} \\
&= \sum_{\mu\le\nu<0}\widehat{\mathcal{Q}}_{M,\nu}(\tau)\begin{pmatrix}\overline{b_\nu}^{(1)}\\ \vdots \\ \overline{b_\nu}^{(p)}\end{pmatrix} \\
&= \widehat{Q}_M(\tau,k,v',\rho').
\end{aligned} \tag{2.112}$$

We are now interested in studying the relationship between $\widehat{Q}_S(\tau,k,v',\rho')$ and $Q_S(\tau,k,v,\rho)$. As discussed before, if $Q_S(\tau,k,v,\rho)=0$ then $F(\tau)$ is a vector-valued modular form on $\Gamma(1)$ of weight $-k$, multiplier system v and representation ρ. Also, it is trivial that if $F(\tau)$ is a modular form then $Q_S(\tau,k,v,\rho)=0$. Analogously we see that if $\widehat{Q}_S(\tau,k,v',\rho')=0$, then $\widehat{F}(\tau)$ is a modular form on $\Gamma(1)$ of weight $-k$, multiplier system v' and representation ρ', and vice versa.

By the definition of $\widehat{Q}_S(\tau,k,v',\rho')$ (2.112), we have that

$$\widehat{Q}_S(\tau,k,v',\rho') = \sum_{\mu\le\nu<0}\widehat{\mathcal{Q}}_{S,\nu}(\tau)\begin{pmatrix}\overline{b_\nu}^{(1)}\\ \vdots \\ \overline{b_\nu}^{(p)}\end{pmatrix},$$

and by (2.107) we can rewrite $\widehat{Q}_S(\tau,k,v',\rho')$ as

$$\begin{aligned}
\widehat{Q}_S(\tau,k,v',\rho') &= \left(\sum_{\mu\le\nu<0}\overline{\mathcal{Q}_{S,\nu}(\overline{\tau})} - 2\overline{\mathcal{Y}_{S,\nu}(\overline{\tau})}\right)\begin{pmatrix}\overline{b_\nu}^{(1)}\\ \vdots \\ \overline{b_\nu}^{(p)}\end{pmatrix} \\
&= \overline{Q_S(\overline{\tau},k,v,\rho)} - 2\sum_{\mu\le\nu<0}\overline{\mathcal{Y}_{S,\nu}(\overline{\tau})}\begin{pmatrix}\overline{b_\nu}^{(1)}\\ \vdots \\ \overline{b_\nu}^{(p)}\end{pmatrix}.
\end{aligned} \tag{2.113}$$

Now we have that if $F(\tau)$ is a modular form then

$$Q_S^{(j,l)}(\tau,k,v,\rho)=\overline{Q_S^{(j,l)}(\overline{\tau},k,v,\rho)}=0, \tag{2.114}$$

and by(2.113), we have that

$$\widehat{Q}_S(\tau,k,v',\rho')=-2\sum_{\mu\le\nu<0}\overline{\mathcal{Y}_{S,\nu}(\overline{\tau})}\begin{pmatrix}\overline{b_\nu}^{(1)}\\ \vdots\\ \overline{b_\nu}^{(p)}\end{pmatrix}. \tag{2.115}$$

On the other hand, we have that if $\widehat{Q}_S(\tau,k,v',\rho')$ is given by the equation (2.115), then we have that

$$Q_S^{(j,l)}(\tau,k,v,\rho)=\overline{Q_S^{(j,l)}(\overline{\tau},k,v,\rho)}=0. \tag{2.116}$$

We have already shown that $Q_T^{(j,l)}(\tau,k,v,\rho)=0$. On the other hand , for M_1, $M_2\in\Gamma(1)$, such that for

$$M_1=\begin{pmatrix}* & *\\ \gamma_1 & \delta_2\end{pmatrix},\quad M_2=\begin{pmatrix}* & *\\ \gamma_2 & \delta_2\end{pmatrix}\quad\text{and}\quad M_3=M_1M_2=\begin{pmatrix}* & *\\ \gamma_3 & \delta_3\end{pmatrix}, \tag{2.117}$$

we have that

$$\begin{aligned}
&Q_{M_1M_2}(\tau,k,v,\rho)\\
&=F(\tau)-v^{-1}(M_1M_2)\rho^{-1}(M_1M_2)\left(\gamma_3\tau+\delta_3\right)^k F(M_1M_2\tau)\\
&=v^{-1}(M_2)\rho^{-1}(M_2)\left(\gamma_2\tau+\delta_2\right)^k\Big(F(M_2\tau)-v^{-1}(M_1)\rho^{-1}(M_1)\left(\gamma_1M_2\tau+\delta_1\right)^k F(M_1M_2\tau)\Big)\\
&\qquad+F(\tau)-v^{-1}(M_2)\rho^{-1}(M_2)\left(\gamma_2\tau+\delta_2\right)^k F(M_2\tau)\\
&=v^{-1}(M_2)\rho^{-1}(M_2)\left(\gamma_2\tau+\delta_2\right)^k Q_{M_1}(M_2\tau,k,v,\rho)+Q_{M_2}(\tau,k,v,\rho)\\
&=Q_{M_2}(\tau,k,v,\rho)+Q_{M_1}\mid_{k,v,\rho}M_2(\tau),
\end{aligned} \tag{2.118}$$

where the slash operator $|_{k,v,\rho}$ was defined in (1.3). Thus, by (2.118), we have that for every $M\in\Gamma(1)$

$$Q_M^{(j,l)}(\tau,k,v,\rho)=0, \tag{2.119}$$

and therefore $F(\tau)$ is a vector-valued modular form on $\Gamma(1)$ of weight $-k$, multiplier system v and representation ρ.

Now we can state the following theorem:

Theorem 2.8 *The function $F(\tau)$ defined in Theorem 2.1 is a vector-valued modular form on $\Gamma(1)$ of weight $-k$, multiplier system v and representation ρ if and only if*

$$\widehat{Q}_S(\tau, k, v', \rho') = -2 \sum_{\mu \le \nu < 0} \overline{\mathcal{Y}_{S,\nu}(\overline{\tau})} \begin{pmatrix} \overline{b_\nu}^{(1)} \\ \vdots \\ \overline{b_\nu}^{(p)} \end{pmatrix}. \tag{2.120}$$

CHAPTER 3

Eichler cohomology and vector valued modular forms

3.1 The cohomology group $H^1_{v,\rho,p}(\Gamma, P_k)$

In Chapter 2 we indicated how to choose the coefficients $b_{-1}^{(j)}, \dots, b_{\mu}^{(j)}$ in formula (2.1) to obtain a vector-valued modular form of negative weight $-k$. However, in general formula (2.1) does not give rise to a vector-valued modular form, since for all $V \in \Gamma$ we have that

$$F(\tau) - F|_{-k,v,\rho}(V\tau) = Q_V(\tau), \tag{3.1}$$

where $Q_V(\tau)$ is a polynomial on τ of degree at most k. The vector polynomials $Q_V(\tau)$ are called period polynomials.

Let $k > 2\delta > 0$, $k \in \mathbb{Z}$, $\mathcal{F}(\Gamma, k+2, v, \rho)$, the space of vector-valued modular forms of dimension p, weight $k+2$ and multiplier system v, with the representation ρ over $\Gamma = \Gamma(1)$, that are holomorphic in $\mathcal{H}$. Let $f(\tau) \in \mathcal{F}(\Gamma, k+2, v, \rho)$, and let $F(\tau)$ be any $(k+1)$-fold indefinite integral of $f(\tau)$. Then, since $F(\tau)$ is differentiable, it satisfies Bol's identity:

$$\begin{aligned} \frac{d^{k+1}}{d\tau^{k+1}}\left((\gamma\tau+\delta)^k F(M\tau)\right) &= (\gamma\tau+\delta)^{-k-2} F^{(k+1)}(M\tau) \\ &= (\gamma\tau+\delta)^{-k-2} f(M\tau), \end{aligned} \tag{3.2}$$

for $\tau \in \mathcal{H}$ and all $M = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$. Therefore,

$$v^{-1}(V)(c\tau + d)^k \rho^{-1}(V) F(V\tau) = F(\tau) + p_V(\tau), \tag{3.3}$$

for $\tau \in \mathcal{H}$ and all $V = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$, where $p_V(\tau)$ is a vector polynomial of degree at most k. The vector polynomials $p_V(\tau)$ are called period (or Eichler) polynomials of $f(\tau)$. We call $F(\tau)$ a vector-valued Eichler integral of weight $-k$ with respect to Γ.

As before, put $(F|_{-k,v,\rho}V)(\tau) = v^{-1}(V)(c\tau + d)^k \rho^{-1}(V)F(V\tau)$. Then we have

$$F|_{-k,v,\rho}V = F + p_V. \tag{3.4}$$

Let $V_1, V_2 \in \Gamma$, $V_3 = V_1V_2$, and $V_i = \begin{pmatrix} * & * \\ c_i & d_i \end{pmatrix}$. Then by (3.3), we see that

$$v^{-1}(V_3)(c_3\tau + d_3)^k \rho^{-1}(V_3)F(V_3\tau) = F(\tau) + p_{V_3}(\tau). \tag{3.5}$$

Also by the consistency condition (1.4) we get that

$$v^{-1}(V_3)(c_3\tau + d_3)^k \rho^{-1}(V_3)F(V_3\tau) \tag{3.6}$$
$$= v^{-1}(V_1)v^{-1}(V_2)(c_1V_2\tau + d_1)^k (c_2\tau + d_2)^k \rho^{-1}(V_3)F(V_3\tau) \tag{3.7}$$
$$= v^{-1}(V_2)(c_2\tau + d_2)^k \rho^{-1}(V_2) \left(v^{-1}(V_1)(c_1V_2\tau + d_1)^k \rho^{-1}(V_1)F(V_3\tau)\right) \tag{3.8}$$
$$= v^{-1}(V_2)(c_2\tau + d_2)^k \rho^{-1}(V_2) \left(F(V_2\tau) + p_{V_1}(V_2\tau)\right) \tag{3.9}$$
$$= F(\tau) + p_{V_2}(\tau) + \left(p_{V_1}|_{-k,v,\rho}V_2\right)(\tau). \tag{3.10}$$

Therefore,

$$p_{V_1V_2}(\tau) = p_{V_3}(\tau) = p_{V_2}(\tau) + \left(p_{V_1}|_{-k,v,\rho}V_2\right)(\tau). \tag{3.11}$$

Now suppose that $\{p_V : V \in \Gamma\}$ is any collection of vector polynomials of degree at most k satisfying (3.11). Then we call $\{p_V : V \in \Gamma\}$ a cocycle. A coboundary is a set $\{p_V : V \in \Gamma\}$ of vector polynomials of degree at most k such that

$$p_V(\tau) = (q|_{-k,v,\rho}V)(\tau) - q(\tau), \tag{3.12}$$

for all $V \in \Gamma$ with $q(\tau)$ a fixed vector polynomial of degree at most k. The cohomology group $H^1_{v,\rho,p}(\Gamma, P_k)$ is defined as the vector space obtained by forming the quotient of the cocycles by the coboundaries, where p is the length of the vector of polynomials and P_k is the vector space of vector polynomials of length p and degree at most k.

Note that given $f(\tau) \in \mathcal{F}(\Gamma, k+2, v, \rho)$, then $F(\tau)$, a $(k+1)$-fold indefinite integral of $f(\tau)$, is determined up to a vector polynomial of degree $\leq k$. Then if we replace $F(\tau)$ by $F(\tau) + q(\tau)$ we find that the cocycle $\{p_V(\tau) : V \in \Gamma\}$ associated to $F(\tau)$ is replaced by the cocycle $\{p^*_V(\tau) : V \in \Gamma\}$, where $p^*_V(\tau) = p_V(\tau) + ((q|_{-k,v,\rho}V)(\tau) - q(\tau))$, so the cocycle $\{p_V(\tau) : V \in \Gamma\}$ is in the same cohomology class as is the cocycle $\{p^*_V(\tau) : V \in \Gamma\}$. Thus $f(\tau) \in \{\Gamma, k+2, v, \rho, p\}$ determines uniquely an element of $H^1_{v,\rho,p}(\Gamma, P_k)$.

3.2 The supplementary function

Knopp and Mason [12] defined the vector-valued Poincaré series $P(\tau, \rho, k, v, \nu, \Gamma, r)$ in the following fashion. Fix ν an integer and r, $1 \leq r \leq p$, and put

$$P(\tau, \rho, k, v, \nu, \Gamma, r) = \frac{1}{2} \sum_{M \in \Gamma} \frac{e^{2\pi i(\nu + m_r)M\tau}}{v(M)(c\tau + d)^k} \rho^{-1}(M) e_r, \tag{3.13}$$

where e_r is the column vector consisting of zeros except for the r^{th} component which is a 1. Here $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ranges over some set of coset representatives for $\Gamma_\infty \backslash \Gamma$.

A normal representation ρ is a representation that satisfies two conditions [12, p.1351]:

1. $\rho(T)$ is diagonal, and

2. $\rho(S^2) = I$.

Let us list some facts about vector-valued Poincaré series. Let ρ be a normal representation of $\Gamma(1)$ and $k > 2\delta$ (1.42). Then,

1. $P(\tau,\rho,k+2,v,\nu,\Gamma,r) \in \mathcal{F}(\Gamma,k+2,\rho,v)$ is a vector-valued modular form of weight $k+2$ [12, p1355].

2. The space of cusp forms $\mathcal{S}(\Gamma,k,\rho,v)$ is spanned by Poincaré series [12, p.1360].

3. The Fourier expansion for a Poincare series $P(\tau) = P(\tau,\rho,k+2,v,\nu,\Gamma,r))$ is given by:
$$P(\tau) = \mathcal{T}_\nu^*(\tau)e_r. \tag{3.14}$$

Here e_r is a column vector of zeros, except for the r^{th} component which is a 1, and $\mathcal{T}_\nu^*(\tau)$ is a matrix defined by

$$\mathcal{T}_\nu^{*(j,l)}(\tau) = \delta_{jl}e^{2\pi i(m_j+\nu)\tau} + \mathcal{R}_\nu^{*(j,l)}(\tau), \tag{3.15}$$

where

$$\mathcal{R}_\nu^{*(j,l)}(\tau) = \sum_{m=0}^{\infty} 2\pi \sum_{c=1}^{\infty} \frac{i^{-k}}{c} A_{c,\nu,m,j,l}\ B^*_{c,\nu,m,j,l}e^{2\pi i(m+m_j)\tau}. \tag{3.16}$$

In (3.16) $B^*_{c,\nu,m,j,l}$ is given by

$$B^*_{c,\nu,m,j,l} = \begin{cases} \left(\frac{-\nu-m_l}{m+m_j}\right)^{-\frac{k+1}{2}} I_{k+1}\left(\frac{4\pi}{c}(-\nu-m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right), & \nu+m_l<0 \\ \frac{1}{(k+1)!}\left(\frac{2\pi(m+m_j)}{c}\right)^{k+1}, & \nu=m_l=0 \\ \left(\frac{\nu+m_l}{m+m_j}\right)^{-\frac{k+1}{2}} J_{k+1}\left(\frac{4\pi}{c}(\nu+m_l)^{\frac{1}{2}}(m+m_j)^{\frac{1}{2}}\right), & \nu+m_l>0, \end{cases} \tag{3.17}$$

with $I_{k+1}(z)$ defined by (1.109) and $J_{k+1}(z)$ by

$$J_{k+1}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{z}{2}\right)^{2n+k+1}}{n!(n+k+1)!}, \quad z \in \mathbb{R} \tag{3.18}$$

[12, pp. 1355-1356].

Suppose that $g(\tau) \in S(\Gamma,k+2,\rho,v)$, and let $s = \dim S(\Gamma,k+2,\rho,v)$. Then there exist complex numbers $b_1,\ldots,b_s$, along with a set of positive integers

$\nu_1, \ldots, \nu_s$, not necessarily distinct, and a set of positive integers $r_1, \ldots, r_s$, where $1 \leq r_j \leq p$, such that

$$g(\tau) = \sum_{i=1}^{s} b_i P(\tau, \rho, k+2, \upsilon, \nu_i, \Gamma, r_i). \tag{3.19}$$

Now we are going to define the function supplementary to a cusp form in a way similar to the definition in Section 2.3:

$$\begin{aligned} \widehat{\nu} &= -1 - \nu \quad \text{if } m_j > 0, \\ \widehat{\nu} &= -\nu \qquad \text{if } m_j = 0, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \widehat{m}_j &= 1 - m_j \quad \text{if } m_j > 0, \\ \widehat{m}_j &= -m_j \qquad \text{if } m_j = 0. \end{aligned} \tag{3.21}$$

Also as we did in Section 2.3, let

$$\widehat{\upsilon}(M) = \overline{\upsilon(M)}, \quad \widehat{\rho}(M) = \overline{\rho(M)} \tag{3.22}$$

and define $\widehat{g}(\tau)$, the function supplementary to $g(\tau)$ as

$$\widehat{g}(\tau) = \sum_{i=1}^{s} \overline{b_i} P(\tau, \widehat{\rho}, k+2, \widehat{\upsilon}, \widehat{\nu_i}, \Gamma, r_i). \tag{3.23}$$

Note that, although the values for $\widehat{\nu}$, $\widehat{m}_j$, $\widehat{\upsilon}$ and $\widehat{\rho}$ are the same as ν', m_j', υ' and ρ' in Section 2.3. Here we form a nonentire vector-valued modular form given a cusp form, while in Section 2.3 we formed an Eichler integral given an Eichler integral.

Now let $\widehat{G}(\tau)$ be the $(k+1)$-fold integral of $\widehat{g}(\tau)$, defined by integrating term-by-term in the expansion at $i\infty$. Note that $\widehat{G}(\tau)$ is the $(k+1)$-fold integral of the function

$$\sum_{i=1}^{s} \mathcal{T}^{*}_{\widehat{\nu}_i}(\tau) e_{r_i} \overline{b_i}, \tag{3.24}$$

normalized so that

$$\widehat{G}(T\tau) = \widehat{\upsilon}(T)\widehat{\rho}(T)\widehat{G}(\tau) = \begin{pmatrix} e^{2\pi i \widehat{m}_1} & & \\ & \ddots & \\ & & e^{2\pi i \widehat{m}_p} \end{pmatrix} \widehat{G}(\tau), \tag{3.25}$$

with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Also we have

$$\widehat{v}^{\,-1}(V)(c\tau+d)^k \widehat{\rho}^{\,-1}(V)\widehat{G}(V\tau) = \widehat{G}(\tau) + \widehat{p}_V(\tau), \tag{3.26}$$

for all $V \in \Gamma$, where $\widehat{p}_V(\tau)$ is a vector polynomial of degree at most k. Now let $G(\tau)$ be the $(k+1)$-fold integral of $g(\tau)$, defined by integrating term-by-term in the expansion at $i\infty$ as we did before with $\widehat{g}(\tau)$. We see that

$$v^{-1}(V)(c\tau+d)^k \rho^{-1}(V)G(V\tau) = G(\tau) + p_V(\tau), \tag{3.27}$$

for all $V \in \Gamma$.

Now we want to consider the relationship between $p_V(\tau)$ and $\widehat{p}_V(\tau)$. To do so we take the same steps as we did in Chapter 2. Let $\mathbf{R}^{*(j,l)}_{\widehat{\nu}}(\tau)$ be defined as the (k+1)-fold integral of $\mathcal{R}^{*(j,l)}_{\widehat{\nu}}(\tau)$. Therefore after changing the order of the summation as we did in (2.25), we have that

$$\mathbf{R}^{*(j,l)}_{\widehat{\nu}}(\tau) = \frac{(2\pi)^{-k}}{i^{2k+1}} \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\substack{c,\,d \\ 0 \le -d < c \\ (c,\,d)\,=\,1}} \widehat{v}^{\,-1}(V)\widehat{x}^{(j,l)} e^{2\pi i a \frac{\widehat{m}_l + \widehat{\nu}}{c}} \sum_{m=0}^{\infty} \frac{B^*_{c,\widehat{\nu},m,j,l} e^{2\pi i (m+\widehat{m}_j)(\tau+\frac{d}{c})}}{(m+\widehat{m}_j)^{k+1}}. \tag{3.28}$$

Applying the Lipschitz summation formula (2.27), as we did in Lemma 2.4, we get that

$$\begin{aligned}
&\frac{(2\pi)^{-k}}{ci^{2k+1}} \sum_{m=0}^{\infty} \frac{B^*_{c,\widehat{\nu},m,j,l} e^{2\pi i (m+\widehat{m}_j)(\tau+\frac{d}{c})}}{(m+\widehat{m}_j)^{k+1}} \\
&= \frac{(2\pi)^{-k}}{ci^{2k+1}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\widehat{\nu}-\widehat{m}_l)^n (m+\widehat{m}_j)^n}{n!(n+k+1)!} e^{2\pi i (m+\widehat{m}_j)(\tau+\frac{d}{c})} \\
&= \frac{(2\pi)^{-k}}{ci^{2k+1}} \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\widehat{\nu}-\widehat{m}_l)^n}{n!(n+k+1)!} \sum_{m=0}^{\infty} (m+\widehat{m}_j)^n e^{2\pi i (m+\widehat{m}_j)(\tau+\frac{d}{c})} \\
&= \frac{(2\pi)^{-k}}{ci^{2k+1}} \sum_{n=0}^{\infty} \left(\frac{2\pi}{c}\right)^{2n+k+1} \frac{(-\widehat{\nu}-\widehat{m}_l)^n}{n!(n+k+1)!} \frac{\Gamma(n+1)}{(2\pi)^{n+1}} \sum_{q=-\infty}^{\infty} e^{2\pi i q \widehat{m}_j} \left(-i\left(\tau+\frac{d}{c}-q\right)\right)^{-n-1} \\
&= \frac{(2\pi)^{-k-1}}{i^{3k+1}} \sum_{q=-\infty}^{\infty} \frac{e^{2\pi i q \widehat{m}_j}\,(c\tau+d-cq)^k}{(-\widehat{\nu}-\widehat{m}_l)^{k+1}} \sum_{n=k+1}^{\infty} \frac{1}{n!} \left(\frac{2\pi i(-\widehat{\nu}-\widehat{m}_l)}{c(c\tau+d-cq)}\right)^n .
\end{aligned} \tag{3.29}$$

(Note that in this case we do not get a different formula if $m = m_j = 0$, because $P(\tau, \rho, k+2, \upsilon, \nu, \Gamma, r)$ is a cusp form. Thus either the the coefficient for $m = 0$ is zero, or $m_j > 0$ and therefore if we follow the same steps as we did in Chapter 2, we get a result similar to (2.107). The difference here is that we do not have the term $2\overline{\mathcal{Y}_{M,\nu}^{(j,l)}(\overline{\tau})}$ (2.36, 2.82), that comes from applying the Lipschitz summation formula.) Therefore we have that

$$\widehat{p}_V(\tau) = \overline{p_V(\overline{\tau})}, \text{ for all } V \in \Gamma. \tag{3.30}$$

Now following Husseini-Knopp [4] we can state the following theorem which is proved exactly in the same way as Theorem 3 in [4].

Theorem 3.1 *Let $k > 2\delta$, $g(\tau) \in \mathcal{S}(\Gamma, k+2, \rho, \upsilon)$ and $\widehat{G}(\tau)$ the function supplementary to $g(\tau)$. Then $g(\tau) \equiv 0$ if and only if $\widehat{G}(\tau) \in \mathcal{F}(\Gamma, -k, \widehat{\rho}, \widehat{\upsilon})$.*

Proof 3.1

If $g(\tau) \equiv 0$, then its (k+1)-fold integral $G(\tau)$ is also identically zero, and therefore $p_V(\tau) = 0$ for all $V \in \Gamma$. Thus by (3.30), we have that $\widehat{p}_V(\tau) = 0$, and therefore

$$\widehat{\upsilon}^{\,-1}(V)(c\tau + d)^k \widehat{\rho}^{\,-1}(V)\widehat{G}(V\tau) = \widehat{G}(\tau). \tag{3.31}$$

Also, since $\widehat{g}(\tau) \in \mathcal{F}(\Gamma, k+2, \widehat{\rho}, \widehat{\upsilon})$, we have that $\widehat{g}(\tau)$ is holomorphic on $\mathcal{H}$ and meromorphic at $i\infty$. Therefore $\widehat{G}(\tau) \in \mathcal{F}(\Gamma, -k, \widehat{\rho}, \widehat{\upsilon})$.

On the other hand, if $\widehat{G}(\tau) \in \mathcal{F}(\Gamma, -k, \widehat{\rho}, \widehat{\upsilon})$, then $\widehat{p}_V(\tau) = 0$ for all $V \in \Gamma$, and by (3.30), we have that $p_V(\tau) = 0$ for all $V \in \Gamma$, and therefore $\widehat{G}(\tau) \in \mathcal{F}(\Gamma, -k, \rho, \upsilon)$. Now since $g(\tau)$ is a cusp form, then it is holomorphic on $\mathcal{H}$ and at $i\infty$, and so is its (k+1)-fold integral $G(\tau)$. By Lemma 4.1 in [11], we have that $G(\tau) \equiv 0$.

3.3 Eichler cohomology and a mapping of $H^1_{\upsilon,\rho,\underline{p}}(\Gamma, P_k)$

Definition 3.1 *A parabolic cocycle $\{p_V : V \in \Gamma\}$ is any collection of vector polynomials of degree at most k and length p satisfying (3.11), in which for* *every parabolic class $\mathcal{B}$ in Γ there exists a fixed polynomial $p_B(\tau)$ of degree $\leq k$ such that*

$$Q_B(\tau) = p_B \mid_{-k,v,\rho} B(\tau) - p_B(\tau), \quad \forall B \in \mathcal{B}. \tag{3.32}$$

(Note that coboundaries are parabolic cocycles.)

Definition 3.2 *The parabolic cohomology group $\widetilde{H}^1_{v,\rho,p}(\Gamma, P_k)$ is defined as the vector space obtained by forming the quotient of the parabolic cocycles by the coboundaries.*

In $\Gamma(1)$ the only parabolic class is the class of T. Note that in $\widetilde{H}^1_{v,\rho,p}(\Gamma, P_k)$ we can always find a cocycle in which $Q_T(\tau) = 0$. For, suppose that $Q_T(\tau) \neq 0$; then there exists a polynomial $p_T(\tau)$ such that (3.32) with $B = T$. Therefore the following polynomial is also in the same cohomology class:

$$\begin{aligned} Q^*_T(\tau) &= Q_T(\tau) - (p_T \mid_{-k,v,\rho} T(\tau) - p_T(\tau)) \\ &= 0. \end{aligned} \tag{3.33}$$

Theorem 3.2 *Let k a positive integer such that $k > 2\delta$, v a multiplier system in weight k and ρ a normal representation of $\Gamma = \Gamma(1)$. Then,*

$$\mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v}) \oplus \mathcal{S}(\Gamma, k+2, \rho, v) \cong \widetilde{H}^1_{v,\rho,p}(\Gamma, P_k), \tag{3.34}$$

with the same mapping as in Theorem 3.1.

Theorem 3.3 *Let k a positive integer, such that $k > 2\alpha$, v a multiplier system in weight k and ρ a normal representation of $\Gamma = \Gamma(1)$. Then,*

$$\mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v}) \oplus \mathcal{M}(\Gamma, k+2, \rho, v) \cong H^1_{v,\rho,p}(\Gamma, P_k), \tag{3.35}$$

and the construction of the mapping is independent of Γ, k, v and ρ. Moreover the map is the same as in Theorem 3.2.

Following Husseini and Knopp [4], we define the mapping

$$\mu : \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v}) \oplus \mathcal{M}(\Gamma, k+2, \rho, v) \to H^1_{v,\rho,p}(\Gamma, P_k) \tag{3.36}$$

by

$$\mu(g(\tau), f(\tau)) = \alpha(g(\tau)) + \beta(f(\tau)), \tag{3.37}$$

where $\beta(f)$ is the cohomology class of the cocycle of the the vector of period polynomials $\{p_V : V \in \Gamma\}$ of the $(k+1)$-fold integral $F(\tau)$ of $f(\tau)$, while $\alpha(g)$ is the cohomology class of the cocycle of the vector of period polynomials $\{q_V : V \in \Gamma\}$ of $\widehat{G}(\tau)$, the $(k+1)$-fold integral of the supplementary function of $g(\tau)$.

Remark 3.1 *To avoid confusion note that*

1. $g(\tau) \in \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v})$,
2. $\widehat{g}(\tau) \in \mathcal{F}(\Gamma, k+2, \rho, v)$,
3. $G(\tau)$ *is the* $(k+1)$*-fold integral of* $g(\tau)$,
4. $\{\widehat{q}_V : V \in \Gamma\}$ *is the cocycle of vector polynomials corresponding to* $G(\tau)$,
5. $\widehat{G}(\tau)$ *is the* $(k+1)$*-fold integral of* $\widehat{g}(\tau)$ *and*
6. $\{q_V : V \in \Gamma\}$ *is the cocycle of vector polynomials corresponding to* $\widehat{G}(\tau)$.

It may seem that the map α depends on the choice of the basis for $\mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v})$. However our mapping is in fact independent of this choice, since the periods, $\{\widehat{q}_V : V \in \Gamma\}$, of the $(k+1)$-fold integral of $g \in \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v})$, $G(\tau)$, are related to those of $\widehat{G}(\tau)$, the $(k+1)$-fold integral of $\widehat{g}(\tau)$, the function supplementary to $g(\tau)$, by (3.30):

$$q_V(\tau) = \overline{\widehat{q}_V(\overline{\tau})}, \tag{3.38}$$

regardless of the choice of the basis.

To show that the map is 1-1 it is enough to show that the kernel of μ is $(0,0)$. Suppose that $\mu(g, f) = 0$. Then there exists a vector polynomial $p(\tau)$, of degree smaller than or equal to k, such that $F(\tau) + \widehat{G}(\tau) + p(\tau) \in \mathcal{F}(\Gamma, -k, \rho, v)$. This is holomorphic on $\mathcal{H}$, and since $F(\tau)$ is the $(k+1)$-fold integral of $f(\tau) \in \mathcal{M}(\Gamma, k+2, \rho, v)$ and $p(\tau)$ is a vector polynomial, we have that the principal part of $F(\tau) + \widehat{G}(\tau) + p(\tau)$ is the same as the principal part of $\widehat{G}(\tau)$, the $(k+1)$-fold integral of $\widehat{g}(\tau)$. This principal part is

$$\sum_{i=1}^{s} \frac{\overline{b_i}}{(-2\pi i(\nu_i + m_{r_i}))^{k+1}} q^{-(\nu_i + m_{r_i})} e_{r_i} \;. \tag{3.39}$$

Since $k+2 > 2+2\delta$, the Fourier coefficients of $F(\tau)+\widehat{G}(\tau)+p(\tau)$ are given by the formula (2.2) and when we apply it to (3.39) we get the Fourier coefficients of $\widehat{G}(\tau)$ as stated by Knopp and Mason [12, Theorem 3.2]. Therefore $\widehat{G}(\tau) = F(\tau) + \widehat{G}(\tau) + p(\tau)$, so $F(\tau) = -p(\tau)$. Since the degree of $p(\tau)$ is less than or equal to k, the $(k+1)^{th}$ derivative is zero, and therefore $f(\tau) \equiv 0$. Also since $\widehat{G}(\tau) \in \mathcal{F}(\Gamma, -k, \rho, v)$, we have that $g(\tau) \equiv 0$ by Theorem 3.1. Thus μ is 1-1.

Now let $\widetilde{\mu}$ be the map μ, restricted to the space of cusp forms $\mathcal{S}(\Gamma, k+2, \rho, v)$. We want to show that

$$\widetilde{\mu} : \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v}) \oplus \mathcal{S}(\Gamma, k+2, \rho, v) \to \widetilde{H}^1_{v,\rho,p}(\Gamma, P_k). \tag{3.40}$$

Let $f(\tau) \in \mathcal{S}(\Gamma, k+2, \rho, v)$, then

$$f^{(j)}(\tau) = \sum_{m+m_j>0} a_m^{(j)} e^{2\pi i(m+m_j)\tau}, \quad \text{for} \;\; 1 \le j \le p, \tag{3.41}$$

and the $(k+1)$-fold integral is

$$F^{(j)}(\tau) = \sum_{m+m_j>0} \frac{a_m^{(j)}}{(2\pi i(m+m_j))^{k+1}} e^{2\pi i(m+m_j)\tau} + p^{(j)}(\tau), \quad \text{for} \;\; 1 \le j \le p. \tag{3.42}$$

Here $p^{(j)}(\tau)$ is a polynomial of degree at most k. We saw at the end of Section 3.1 that the cohomology class of $F(\tau)$ is the same as the cohomology class of $F(\tau) - p(\tau)$, so we can assume without loss of generality that $p^{(j)}(\tau) = 0$. It is clear that if $p(\tau) = 0$, we have that

$$F \mid_{-k,v,\rho} T(\tau) = F(\tau), \tag{3.43}$$

which implies that $p_T(\tau) = 0$. Therefore $\beta(f) \in \widetilde{H}^1_{v,\rho,p}(\Gamma, P_k)$. On the other hand if $g \in \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v})$, then

$$g(\tau) = \sum_{i=1}^{s} b_i P(\tau, \widehat{\rho}, k+2, \widehat{v}, \widehat{\nu_i}, \Gamma, r_i), \tag{3.44}$$

and

$$\widehat{g}(\tau) = \sum_{i=1}^{s} \overline{b_i} P(\tau, \rho, k+2, v, \nu_i, \Gamma, r_i). \tag{3.45}$$

In all the expansions of the Poincare series $P(\tau, \widehat{\rho}, k+2, \widehat{v}, \widehat{\nu_i}, \Gamma, r_i) \in \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v})$, we have that $m + \widehat{m}_j > 0$, for all $1 \leq j \leq p$ and all $1 \leq i \leq s$. Therefore in the expansions of $P(\tau, \rho, k+2, v, \nu_i, \Gamma, r_i)$ we will also have that $m + m_j > 0$, for all $1 \leq j \leq p$ and all $1 \leq i \leq s$. Thus

$$\widehat{g}^{(j)}(\tau) = \sum_{m+m_j>0} c_m^{(j)} e^{2\pi i(m+m_j)\tau}, \quad \text{for} \quad 1 \leq j \leq p, \tag{3.46}$$

and the $(k+1)$-fold integral is

$$\widehat{G}^{(j)}(\tau) = \sum_{m+m_j>0} \frac{c_m^{(j)}}{(2\pi i(m+m_j))^{k+1}} e^{2\pi i(m+m_j)\tau} + q^{(j)}(\tau), \quad \text{for} \quad 1 \leq j \leq p. \tag{3.47}$$

Here $q^{(j)}(\tau)$ is a polynomial of degree at most k. We assume without loss of generality that $q^{(j)}(\tau) = 0$. It is clear that if $q(\tau) = 0$, we have that

$$\widehat{G} \mid_{-k,v,\rho} T(\tau) = \widehat{G}(\tau), \tag{3.48}$$

which implies that $q_T(\tau) = 0$. Therefore $\alpha(g) \in \widetilde{H}^1_{v,\rho,p}(\Gamma, P_k)$. Thus

$$\widetilde{\mu}(g(\tau), f(\tau)) = \alpha(g(\tau)) + \beta(f(\tau)) \in \widetilde{H}^1_{v,\rho,p}(\Gamma, P_k). \tag{3.49}$$

Therefore we have shown not only that $\widetilde{\mu}$ maps the given spaces into the Eichler cohomology $\widetilde{H}^1_{v,\rho,p}(\Gamma, P_k)$, but also that $\widetilde{\mu}$ is 1-1, since μ is.

It remains to show that the maps μ and $\widetilde{\mu}$ are onto. We use the vector-valued generalized Poincaré series, to show that $\widetilde{\mu}$ is onto.

3.4 The vector-valued generalized Poincaré series $\Psi(\tau; \{p_V(\tau)\}, r, w)$

Lehner [13] defined a vector-valued generalized Poincaré series. Let $\{Q_V(\tau)\}$ be a parabolic cocycle of vector polynomials of degree $\leq k$ on $\Gamma(1)$, with

$k \in \mathbb{Z}^+$, v a multiplier system in $\Gamma(1)$ and ρ a normal representation. Assume also that $Q_T(\tau) = 0$. We define the vector-valued generalized Poincaré series as

$$\Psi^{(j)}(\tau; r) = \sum_{V \in \mathcal{L}} \frac{Q_V^{(j)}(\tau)}{(c\tau + d)^r}, \quad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.50}$$

where r is a large positive even integer and $\mathcal{L}$ is any set in $\Gamma(1)$ containing all transformations with different lower rows. Now we note that if M and M^* have the same lower row, then we can write $M = T^l M^*$ and therefore

$$\begin{aligned} Q_M(\tau) &= Q_{T^l M^*}(\tau) \\ &= Q_{T^l} \mid_{-k,v,\rho} M^*(\tau) + Q_{M^*}(\tau) \\ &= Q_{M^*}(\tau), \end{aligned} \tag{3.51}$$

so that $\Psi^{(j)}(\tau; r)$ does not depend on the choice of coset representatives.

To study the convergence of $\Psi(\tau; r)$, we need the following facts:

1. Lemma 4 in [9]: For real numbers c, d and $\tau = x + iy$, we have

$$\frac{y^2}{1 + 4|\tau|^2}(c^2 + d^2) \leq |c\tau + d|^2 \leq 2(|\tau|^2 + y^{-2})(c^2 + d^2). \tag{3.52}$$

2. Let $a_0^{(j)}, \ldots, a_k^{(j)}$ be the coefficients of $Q_S^{(j)}(\tau)$. Then if

$$K_1 = \sum_{j=1}^{p} \sum_{t=0}^{k} \left| a_t^{(j)} \right|, \tag{3.53}$$

we have that

$$\left| Q_S^{(j)}(\tau) \right| < K_1 \left(|\tau|^k + y^{-1} \right), \quad \text{for } \tau \in \mathcal{H} \quad \text{and} \quad 1 \leq j \leq p. \tag{3.54}$$

3. Since $Q_T(\tau) = 0$, we have that

$$\begin{aligned} Q_{T^m}(\tau) &= Q_T \mid_{-k,v,\rho} T^{m-1}(\tau) + Q_T \mid_{-k,v,\rho} T^{m-2}(\tau) + \ldots + Q_T(\tau) \\ &= 0. \end{aligned} \tag{3.55}$$

4. Since $Q_{T^m}(\tau) = 0$, we have that

$$\begin{aligned} Q_{T^m V}(\tau) &= Q_{T^m} \mid_{-k,v,\rho} V(\tau) + Q_V(\tau) \\ &= Q_V(\tau). \end{aligned} \tag{3.56}$$

5. Let $q_1, \ldots, q_s \in \mathbb{Z}^+$, $n \in \mathbb{Z}$,

$$V = ST^{(-1)^s q_s} ST^{(-1)^{s-1} q_{s-1}} S \ldots T^{q_2} ST^{-q_1} ST^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.57}$$

and for $0 \leq j \leq s$ define the matrix

$$M_j = T^{(-1)^j q_j} ST^{(-1)^{j-1} q_{j-1}} S \ldots T^{q_2} ST^{-q_1} ST^n = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}, \tag{3.58}$$

with $M_0 = T^n$ and $SM_s = V$, then

$$|c| \geq |\alpha_j|, \quad |c| \geq |\gamma_j|, \quad |d| \geq |\beta_j| \quad \text{and} \quad |d| \geq |\delta_j|. \tag{3.59}$$

6. For $V \in \Gamma(1)$, by (3.11) we have that

$$\begin{aligned} Q_{-V}(\tau) &= Q_{-I}(\tau) + Q_V \mid_{-k,v,\rho} (-I(\tau)) \\ &= v^{-1}(-I)\rho^{-1}(-I)(-1)^k Q_V(\tau) + Q_{-I}(\tau) \\ &= Q_V(\tau). \end{aligned} \tag{3.60}$$

7. Finally,

$$Q_I(\tau) = Q_{-I}(\tau) = 0. \tag{3.61}$$

Note that by (3.55), we can rewrite (3.50) as

$$\Psi^{(j)}(\tau; r) = \sum_{\substack{V \in \mathcal{L} \\ c \neq 0}} \frac{Q_V^{(j)}(\tau)}{(c\tau + d)^r}, \quad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.62}$$

Now we want to find a bound for the cocycle $Q_V(\tau)$ for $V \in \Gamma(1)$. By (1.37), we have that

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm T^m ST^{(-1)^s q_s} ST^{(-1)^{s-1} q_{s-1}} S \ldots T^{q_2} ST^{-q_1} ST^n. \tag{3.63}$$

Recall (Lemma 1.1) that s is the number of steps in the Euclidean algorithm applied to the pair c, d.

From the definition of a cocycle and by (3.55), (3.11), (1.37) and (3.60) we have that

$$\begin{aligned}
Q_V(\tau) &= Q_{\pm T^m ST^{(-1)^s q_s} ST^{(-1)^{s-1} q_{s-1}} S \dots T^{q_2} ST^{-q_1} ST^n}(\tau) \\
&= Q_{ST^{(-1)^s q_s} ST^{(-1)^{s-1} q_{s-1}} S \dots T^{q_2} ST^{-q_1} ST^n}(\tau) \\
&= Q_{SM_s}(\tau) \\
&= Q_S \mid_{-k,v,\rho} M_s(\tau) + Q_{M_s}(\tau) \\
&= Q_S \mid_{-k,v,\rho} M_s(\tau) + Q_{T^{(-1)^s q_s} SM_{s-1}}(\tau) \\
&= Q_S \mid_{-k,v,\rho} M_s(\tau) + Q_{SM_{s-1}}(\tau) \\
&= Q_S \mid_{-k,v,\rho} M_s(\tau) + Q_S \mid_{-k,v,\rho} M_{s-1}(\tau) + Q_{M_{s-1}}(\tau) \\
&= Q_S \mid_{-k,v,\rho} M_s(\tau) + \dots + Q_S \mid_{-k,v,\rho} M_1(\tau) + Q_S \mid_{-k,v,\rho} M_0(\tau) + Q_{M_0}(\tau) \\
&= \sum_{h=0}^{s} Q_S \mid_{-k,v,\rho} M_h(\tau) + Q_{T^n}(\tau) \\
&= \sum_{h=0}^{s} Q_S \mid_{-k,v,\rho} M_h(\tau) \\
&= \sum_{h=0}^{s} v^{-1}(M_h)\rho^{-1}(M_h)(\gamma_h \tau + \delta_h)^k Q_S(M_h \tau).
\end{aligned} \tag{3.64}$$

Now by (1.42) and the fact that $|\gamma_h| \leq |c|$ (3.59), we get

$$\begin{aligned}
\left| Q_V^{(j)}(\tau) \right| &\leq \sum_{l=1}^{p} \sum_{h=0}^{s} \left| (\rho^{-1}(M_h))^{(j,l)} \right| |\gamma_h \tau + \delta_h|^k \mid Q_S^{(l)}(M_h \tau) \mid \\
&\leq K_1^* \sum_{l=1}^{p} \sum_{h=0}^{s} |\gamma_h|^{2\delta} |\gamma_h \tau + \delta_h|^k \mid Q_S^{(l)}(M_h \tau) \mid \\
&\leq K_1^* c^{2\delta} \sum_{l=1}^{p} \sum_{h=0}^{s} |\gamma_h \tau + \delta_h|^k \mid Q_S^{(l)}(M_h \tau) \mid,
\end{aligned} \tag{3.65}$$

and from (3.54) we obtain

$$\begin{aligned}
|\gamma_h \tau + \delta_h|^k \left| Q_S^{(l)}(M_h \tau) \right| &\leq K_2^* |\gamma_h \tau + \delta_h|^k \left(|M_h \tau|^k + y^{-1} |\gamma_h \tau + \delta_h|^2 \right) \\
&= K_2^* \left(|\alpha_h \tau + \beta_h|^k + y^{-1} |\gamma_h \tau + \delta_h|^{k+2} \right).
\end{aligned} \tag{3.66}$$

By (3.52) we have that

$$|\alpha_h\tau+\beta_h|^k \le 2^{\frac{k}{2}}\left(|\tau|^2+y^{-2}\right)^{\frac{k}{2}}\left(\alpha_h^2+\beta_h^2\right)^{\frac{k}{2}} \tag{3.67}$$

and

$$|\gamma_h\tau+\delta_h|^{k+2} \le 2^{\frac{k+2}{2}}\left(|\tau|^2+y^{-2}\right)^{\frac{k+2}{2}}\left(\gamma_h^2+\delta_h^2\right)^{\frac{k+2}{2}}. \tag{3.68}$$

Thus, (3.66) becomes

$$|\gamma_h\tau+\delta_h|^k\left|Q_S^{(l)}(M_h\tau)\right| \le K_3^*\left(|\tau|^2+y^{-2}\right)^{\frac{k+2}{2}}\left(\left(\alpha_h^2+\beta_h^2\right)^{\frac{k}{2}}+y^{-1}\left(\gamma_h^2+\delta_h^2\right)^{\frac{k+2}{2}}\right), \tag{3.69}$$

and by (3.59) we have that

$$\begin{aligned}|\gamma_h\tau+\delta_h|^k\left|Q_S^{(l)}(M_h\tau)\right| &\le K_3^*\left(|\tau|^2+y^{-2}\right)^{\frac{k+2}{2}}\left(\left(c^2+d^2\right)^{\frac{k}{2}}+y^{-1}\left(c^2+d^2\right)^{\frac{k+2}{2}}\right)\\ &\le K_3^*\left(c^2+d^2\right)^{\frac{k+2}{2}}\left(|\tau|^2+y^{-2}\right)^{\frac{k+2}{2}}\left(1+y^{-1}\right).\end{aligned} \tag{3.70}$$

Therefore we rewrite (3.65) applying (3.70) as

$$\begin{aligned}\left|Q_V^{(j)}(\tau)\right| &\le K_4^*c^{2\delta}\sum_{h=0}^{s}\left(c^2+d^2\right)^{\frac{k+2}{2}}\left(|\tau|^2+y^{-2}\right)^{\frac{k+2}{2}}\left(1+y^{-1}\right)\\ &\le K_4^*(s+1)c^{2\delta}\left(c^2+d^2\right)^{\frac{k+2}{2}}\left(|\tau|^2+y^{-2}\right)^{\frac{k+2}{2}}\left(1+y^{-1}\right).\end{aligned} \tag{3.71}$$

Also, by (1.39) we have that

$$s+1 \le 2\left(c^2+d^2\right) \tag{3.72}$$

and clearly

$$c^{2\delta} \le \left(c^2+d^2\right)^{\delta}. \tag{3.73}$$

Thus, we rewrite (3.71) as

$$\left|Q_V^{(j)}(\tau)\right| \le K_5^*\left(c^2+d^2\right)^{\frac{k+2\delta+4}{2}}\left(|\tau|^2+y^{-2}\right)^{\frac{k+2}{2}}\left(1+y^{-1}\right). \tag{3.74}$$

Now by (3.52) we rewrite (3.74)

$$\begin{aligned}\left|Q_V^{(j)}(\tau)\right| &\le K_5^*\left(|c\tau+d|^2\,\frac{1+4|\tau|^2}{y^2}\right)^{\frac{k+2\delta+4}{2}}\left(|\tau|^2+y^{-2}\right)^{\frac{k+2}{2}}\left(1+y^{-1}\right)\\ &\le \lambda(\tau)\,|c\tau+d|^{k+2\delta+4}.\end{aligned} \tag{3.75}$$

Here,

$$\lambda(\tau) = K_5^* \left(\frac{1+4|\tau|^2}{y^2} \right)^{\frac{k+2\delta+4}{2}} \left(|\tau|^2 + y^{-2} \right)^{\frac{k+2}{2}} \left(1 + y^{-1} \right). \tag{3.76}$$

Thus, by (3.62), (3.75) and (3.76) we have that

$$\left| \Psi^{(j)}(\tau; r) \right| \leq \lambda(\tau) \sum_{\substack{V \in \mathcal{L} \\ c \neq 0}} |c\tau + d|^{k+2\delta+4-r}. \tag{3.77}$$

Now we want to show that $\Psi^{(j)}(\tau; r)$ converges uniformly on

$$\mathcal{S} = \left\{ \tau = x + iy : \ |x| \leq \frac{1}{2}, \ y \geq M > 0 \right\}. \tag{3.78}$$

First we note that for $\tau \in \mathcal{S}$ we can bound the function $\lambda(\tau)$ (3.76),

$$\lambda(\tau) \leq K_1(\mathcal{S})(1 + y^{k+2}), \tag{3.79}$$

where $K_1(\mathcal{S})$ is a constant that depends on $\mathcal{S}$. Also, since $c \neq 0$, we have that

$$|c\tau + d|^{k+2} \geq |c|^{k+2} y^{k+2} > y^{k+2}. \tag{3.80}$$

By (3.79), (3.80) and (3.77) and for $\tau \in \mathcal{S}$ we have

$$\begin{aligned} \left| \Psi^{(j)}(\tau; r) \right| &\leq K_1(\mathcal{S}) \frac{1 + y^{k+2}}{y^{k+2}} \sum_{\substack{V \in \mathcal{L} \\ c \neq 0}} |c\tau + d|^{2k+2\delta+6-r} \\ &\leq K_2(\mathcal{S}) \sum_{V \in \mathcal{L}} |c\tau + d|^{2k+2\delta+6-r}. \end{aligned} \tag{3.81}$$

Since the summation converges absolute-uniformly on $\mathcal{S}$ for $r > 2\delta + 2k + 8$ [1, pp. 15-16], so does $\Psi^{(j)}(\tau; r)$. Therefore $\Psi^{(j)}(\tau; r)$ is holomorphic on $\mathcal{H}$ and at $i\infty$. Moreover we have that

$$\lim_{\tau \to i\infty} \Psi^{(j)}(\tau; r) = \sum_{\substack{V \in \mathcal{L} \\ c \neq 0}} \lim_{\tau \to i\infty} \frac{Q_V^{(j)}(\tau)}{(c\tau + d)^r} = 0, \tag{3.82}$$

since we can put the limit inside the summation by the uniform convergence on $\mathcal{S}$.

Note that for every M in $\Gamma(1)$ there is a one-to-one correspondence between $\mathcal{L}$ and $\mathcal{L}M$. Therefore, $M = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$, $VM = \begin{pmatrix} * & * \\ \widetilde{\gamma} & \widetilde{\delta} \end{pmatrix} \in \Gamma$, and using the absolute convergence and the fact that $r \in 2\mathbb{Z}^+$, we have

$$\begin{aligned}
&\Psi \mid_{-k,v,\rho} M(\tau) && (3.83)\\
&= v^{-1}(M)(\gamma\tau+\delta)^k \rho^{-1}(M)\Psi(M\tau;r) && (3.84)\\
&= \sum_{V\in\mathcal{L}} v^{-1}(M)(\gamma\tau+\delta)^k \rho^{-1}(M) Q_V(M\tau)(cM\tau+d)^{-r} && (3.85)\\
&= \sum_{V\in\mathcal{L}} \left(Q_V \mid_{-k,v,\rho} M(\tau)\right)(cM\tau+d)^{-r} && (3.86)\\
&= \sum_{V\in\mathcal{L}} \left(Q_{VM}(\tau) - Q_M(\tau)\right)(cM\tau+d)^{-r} && (3.87)\\
&= (\gamma\tau+\delta)^r \sum_{VM\in\mathcal{L}} \left(Q_{VM}(\tau) - Q_M(\tau)\right)\left(\widetilde{\gamma}\tau+\widetilde{\delta}\right)^{-r} && (3.88)\\
&= (\gamma\tau+\delta)^r \left(\Psi(\tau;r) - \psi(\tau;r)Q_M(\tau)\right). && (3.89)
\end{aligned}$$

Here $\psi(\tau;r)$ is the classical Eisenstein series

$$\psi(\tau;r) = \sum_{V\in\mathcal{L}} (c\tau+d)^{-r}, \tag{3.90}$$

which converges absolutely for $r > 2$. Also we have the transformation law

$$\psi(M\tau;r) = (\gamma\tau+\delta)^r \psi(\tau;r), \text{ for all } M \in \Gamma, \tag{3.91}$$

provided r is an even integer bigger than 2.

Now we define the vector valued function

$$F(\tau) = -\frac{\Psi(\tau;r)}{\psi(\tau;r)}. \tag{3.92}$$

By (3.89) and (3.91) we get

$$\begin{aligned}
&F \mid_{-k,v,\rho} M(\tau)\\
&= -\frac{\Psi \mid_{-k,v,\rho} M(\tau)}{\psi(M\tau;r)}\\
&= -\frac{\Psi(\tau;r)}{\psi(\tau;r)} + Q_M(\tau)\\
&= F(\tau) + Q_M(\tau).
\end{aligned}$$

Since $\Psi(\tau;r)$ converges absolute-uniformly on compacts of $\mathcal{H}$ for r sufficiently large, it is holomorphic on $\mathcal{H}$ and therefore $\Psi(\tau;r)$ does not have any pole on $\mathcal{H}$.

In order to avoid a pole in $F(\tau)$ at $\tau = i$, and at $\tau = \frac{1\pm\sqrt{3}i}{2}$, we can choose a convenient r. Basically we want to avoid a zero of $\psi(\tau;r)$ at those points. To do so note that by (3.52) for $\tau = i, \frac{1\pm\sqrt{3}i}{2}$ we have that

$$\begin{aligned}\left|\sum_{\substack{V\in\mathcal{L}\\|c|>1}}\frac{1}{(c\tau+d)^r}\right| &\leq \sum_{\substack{V\in\mathcal{L}\\|c|>1}}\left|\frac{1}{(c\tau+d)^r}\right| \\ &\leq 2\sum_{c=2}^{\infty}\sum_{\substack{0\leq d<c\\(c,d)=1}}\sum_{m=-\infty}^{\infty}\left|\frac{1}{(c\tau+cm+d)^r}\right| \\ &\leq K\frac{(1+4|\tau|^2)^{\frac{r}{2}}}{y^r}\sum_{c=2}^{\infty}\sum_{\substack{0\leq d<c\\(c,d)=1}}\sum_{m=-\infty}^{\infty}\frac{1}{c^r\left(1+\left(m+\frac{d}{c}\right)^2\right)^{\frac{r}{2}}} \\ &\leq K\frac{(1+4|\tau|^2)^{\frac{r}{2}}}{y^r}\sum_{c=2}^{\infty}\frac{1}{c^{r-1}}\left(\sum_{m=0}^{\infty}\frac{1}{(m^2+1)^{\frac{r}{2}}}+\sum_{m=-\infty}^{-1}\frac{1}{((m+1)^2+1)^{\frac{r}{2}}}\right) \\ &= 2K\frac{(1+4|\tau|^2)^{\frac{r}{2}}}{y^r}\sum_{c=2}^{\infty}\frac{1}{c^{r-1}}\sum_{m=0}^{\infty}\frac{1}{(m^2+1)^{\frac{r}{2}}}\ ,\end{aligned} \tag{3.93}$$

and we can choose an r large enough to make the above summation less than 1 for $\tau = i, \frac{1\pm\sqrt{3}i}{2}$. We can also choose r large enough so that

$$\left|\sum_{\substack{d=-\infty\\d\neq 0}}^{\infty}\frac{1}{(i+d)^r}\right|<1,\quad \left|\sum_{\substack{d=-\infty\\d\neq 0}}^{\infty}\frac{1}{(-i+d)^r}\right|<1 \tag{3.94}$$

and

$$\left|\sum_{\substack{d=-\infty\\d\neq 0}}^{\infty}\frac{1}{(\frac{1\pm\sqrt{3}i}{2}+d)^r}\right|<1,\quad \left|\sum_{\substack{d=-\infty\\d\neq 0}}^{\infty}\frac{1}{(-\frac{1\pm\sqrt{3}i}{2}+d)^r}\right|<1. \tag{3.95}$$

Now if $r \equiv 0 \pmod{12}$ we have that

$$\psi(i) = 1 + \frac{1}{(-1)^r} + \frac{1}{i^r} + \frac{1}{(-i)^r} + \sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{1}{(i+d)^r} + \sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{1}{(-i+d)^r} + \sum_{\substack{V \in \mathcal{L} \\ |c|>1}} \frac{1}{(ci+d)^r}$$
$$= 4 + \sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{1}{(i+d)^r} + \sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{1}{(-i+d)^r} + \sum_{\substack{V \in \mathcal{L} \\ |c|>1}} \frac{1}{(ci+d)^r}, \tag{3.96}$$

and therefore $|\psi(i)| > 0$. Also ,

$$\psi\left(\frac{1 \pm \sqrt{3}i}{2}\right) = 1 + \frac{1}{(-1)^r} + \frac{1}{\left(\frac{1\pm\sqrt{3}i}{2}\right)^r} + \frac{1}{\left(-\frac{1\pm\sqrt{3}i}{2}\right)^r} + \sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{1}{(\frac{1\pm\sqrt{3}i}{2}+d)^r}$$
$$+ \sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{1}{(-\frac{1\pm\sqrt{3}i}{2}+d)^r} + \sum_{\substack{V \in \mathcal{L} \\ c>1}} \frac{1}{\left(c\left(\frac{1\pm\sqrt{3}i}{2}\right)+d\right)^r}$$
$$= 4 + \sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{1}{(\frac{1\pm\sqrt{3}i}{2}+d)^r} + \sum_{\substack{d=-\infty \\ d\neq 0}}^{\infty} \frac{1}{(-\frac{1\pm\sqrt{3}i}{2}+d)^r}$$
$$+ \sum_{\substack{V \in \mathcal{L} \\ c>1}} \frac{1}{\left(c\left(\frac{1\pm\sqrt{3}i}{2}\right)+d\right)^r}, \tag{3.97}$$

and therefore $\left|\psi\left(\frac{1\pm\sqrt{3}i}{2}\right)\right| > 0$. Thus for r a large integer divisible by 12, we have that $F(\tau)$ does not have a pole at $\tau = i$ or $\frac{1\pm\sqrt{3}i}{2}$.

3.5 Construction of a convenient vector valued modular form of negative weight $-k < -2\delta$ on $\Gamma(1)$

Next we want to modify the function $F(\tau)$ by substracting a form with the same principal part at all poles of $F(\tau)$ on $\mathcal{H}$, so that we get an Eichler integral with poles only at $i\infty$ and the same periods as $F(\tau)$. To do so, we want to construct a function $I_{\tau_0,r}(\tau)$ which has a pole of order s at τ_0 in the fundamental region of $\Gamma(1)$ and possibly at $i\infty$, and which is holomorphic everywhere else on the fundamental region.

In Theorem 2.7 we showed how to construct a vector-valued modular form $H(\tau)$ on $\Gamma(1)$ which is analytic in $\mathcal{H}$. We define $H(\tau)$ as in (2.1), with $k > 2\delta$ and μ sufficiently large.

Now we want to modify the function $H(\tau)$ so that we have a pole of order s at τ_0.in the r^{th} component. Let h the order of the zero of $H(\tau)$ at τ_0 in the r^{th} component. We can consider the function

$$I^{(j)}_{\tau_0,r}(\tau, s) = \begin{cases} H^{(j)}(\tau) & \text{if} \quad j \neq r \\ \frac{H^{(j)}(\tau)}{(J(\tau)-J(\tau_0))^{s+h}} & \text{if} \quad j = r \end{cases} \tag{3.98}$$

It is easy to see that $I_{\tau_0,r}(\tau, s) \in \mathcal{F}(\Gamma, -k, \rho, v)$, since $H(\tau) \in \mathcal{F}(\Gamma, -k, \rho, v)$, and $J(\tau)$ is invariant under the action of $|_{-k,v,\rho}$.

Since $J(\tau)$ attains every complex value only once in the fundamental region, we see that $J(\tau) - J(\tau_0)$ will be zero of order 1 in the usual variable $\tau - \tau_0$, except for $\tau = i, \frac{1\pm\sqrt{3}i}{2}$, where the function $J(\tau) - J(\tau_0)$ does not have a zero of order 1 in the usual variable $\tau - \tau_0$.

The function defined in (3.92)

$$F(\tau) = -\frac{\Psi(\tau)}{\psi(\tau)} \tag{3.99}$$

could have poles at $\mathcal{H}$. Those poles correspond to zeros of $\psi(\tau)$, since we chose r large enough to make $\Psi(\tau)$ converge in $\mathcal{H}$. Also we chose $r \equiv 0 \pmod{12}$, to make sure that $\psi(\tau)$ has no zeros at $\tau = i, \frac{1\pm\sqrt{3}i}{2}$. Therefore $F(\tau)$ will not have a pole at $\tau = i, \frac{1\pm\sqrt{3}i}{2}$. Now if $F(\tau)$ has a pole of at τ_0 at the compoonent r, and the principal part of $F(\tau)$ at the component r is given by

$$\sum_{n_o \leq n < 0} c_n^{(r)} (\tau - \tau_0)^n \tag{3.100}$$

we will construct a form $I_{\tau_0,r}(\tau)$ with the same principal part as $F(\tau)$ in the component. Put

$$I_{\tau_0,r}(\tau) = \sum_{n_o \le n < 0} \frac{c_n^{(r)}}{a(\tau_0, n, r)} I_{\tau_0,r}(\tau, n), \tag{3.101}$$

where $a(\tau_0, n, r)$ is given by

$$\lim_{\tau \to \tau_0} I_{\tau_0,r}^{(r)}(\tau, n)(\tau - \tau_0)^n, \tag{3.102}$$

and $I_{\tau_0,r}^{(r)}(\tau, n)$ is given by (3.98). We note that the principal parts of $F(\tau)$ and $I_{\tau_0,r}(\tau)$ are the same in the component r.

We repeat the process as many times as necessary to construct an automorphic vector-valued form $Q(\tau) \in \mathcal{F}(\Gamma, -k, \rho, v)$, such that both $F(\tau)$ and $Q(\tau)$ have the same poles with the same principal parts on a fundamental region for $\Gamma(1)$. Therefore,

$$F(\tau) - Q(\tau) \tag{3.103}$$

is holomorphic on $\mathcal{H}$ and for all $M \in \Gamma(1)$ we have that

$$(F - Q) \mid_{-k,v,\rho} M(\tau) \tag{3.104}$$

$$= F \mid_{-k,v,\rho} M(\tau) - Q \mid_{-k,v,\rho} M(\tau) \tag{3.105}$$

$$= F(\tau) - Q(\tau) + p_M(\tau). \tag{3.106}$$

$$\tag{3.107}$$

Now let $W(\tau)$ be defined as

$$W(\tau) = \frac{d^{k+1}}{d\tau^{k+1}} \left(F(\tau) - Q(\tau) \right), \tag{3.108}$$

where the derivative can be calculated term-by-term on the expansion at $i\infty$, since it is analytic everywhere else. By the discussion in section 1 we see that $W(\tau) \in \mathcal{F}(\Gamma, k+2, \rho, v)$. Now let $\sum_{\nu_j=-\mu_0}^{-1} b_{\nu_j}^{(j)} e^{2\pi i(\nu_j + m_j)}$ be the principal part of $W^{(j)}(\tau)$ at $i\infty$, where $b_\nu^{(j)}$'s are a complex numbers. Then we can write

$$W(\tau) = \sum_{j=1}^{p} \sum_{\nu_j=-\mu_0}^{-1} b_{\nu_j}^{(j)} P(\tau, \rho, k+2, v, \nu_j, \Gamma, j) + B(\tau), \tag{3.109}$$

where $P(\tau, \rho, k+2, v, \nu_j, \Gamma, j)$ is a Poincare series with $\nu_j + m_j < 0$, and therefore the first term has the same poles as $W(\tau)$. $B(\tau)$ is therefore in $\mathcal{M}(\Gamma, k+2, \rho, v)$. For all j, $\nu_j + m_j < 0$, and thus the function supplementary to $P(\tau, \rho, k+2, v, \nu, \Gamma, i)$ is a cusp form. So let $\beta(B(\tau))$ be the cohomology class of the cocycle of the the period vector polynomials $\{p_V : V \in \Gamma\}$ of the $(k+1)$-fold integral of $B(\tau)$. Now put

$$A(\tau) = \sum_{j=1}^{p} \sum_{\nu=-\mu_0}^{-1} \overline{b_\nu}^{(j)} P(\tau, \widehat{\rho}, k+2, \widehat{v}, \widehat{\nu}_j, \Gamma, j) \in \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v}), \qquad (3.110)$$

and let $\alpha(A(\tau))$ be the cohomology class of the cocycle of the the period vector polynomials $\{q_V : V \in \Gamma\}$ of the $(k+1)$-fold integral of

$$\sum_{j=1}^{p} \sum_{\nu=-\mu_0}^{-1} b_\nu^{(j)} P(\tau, \rho, k+2, v, \nu_j, \Gamma, j). \qquad (3.111)$$

It only remains to show that the function $B(\tau)$ is indeed a cusp form in $\mathcal{S}(\Gamma, k+2, \rho, v)$. We already saw at the end of Section 3.3, that the cohomology class $\alpha(A(\tau))$ is parabolic. We will show that if $B(\tau)$ is not a cusp form, then $\beta(B(\tau))$ will not be parabolic, and therefore $\alpha(A(\tau)) + \beta(B(\tau))$ will not be parabolic, which is a contradiction, since we started with a parabolic cohomology class and showed that $\alpha(A(\tau)) + \beta(B(\tau))$ equals the given parabolic cohomology class.

Suppose that $B(\tau) \in \mathcal{M}(\Gamma, k+2, \rho, v) - \mathcal{S}(\Gamma, k+2, \rho, v)$, then for some j, we have that $m_j = 0$ and

$$B^{(j)}(\tau) = \sum_{m=0}^{\infty} a_m^{(j)} e^{2\pi i m \tau}, \quad a_0^{(j)} \neq 0, \qquad (3.112)$$

and its $(k+1)$-fold integral will be

$$C^{(j)}(\tau) = \sum_{m=0}^{\infty} \frac{a_m^{(j)} e^{2\pi i m \tau}}{2\pi i m} + \frac{a_0^{(j)} \tau^{k+1}}{(k+1)!} + p^{(j)}(\tau), \qquad (3.113)$$

and

$$(C(\tau) - C\,|_{-k,v,\rho}\, T(\tau))^{(j)} = \frac{a_0^{(j)}}{k!} \tau^k + \ldots, \quad \frac{a_0^{(j)}}{k!} \neq 0. \qquad (3.114)$$

Therefore all cocycles of the cohomology class of (3.113) will have a polynomial of degree k, in which the coefficient of τ^k in the j^{th} component will never be zero since the coefficient of τ^k of $p^{(j)}(\tau)$ cancels with the coefficent of τ^k of $p^{(j)}(\tau+1)$. And therefore $\beta(B(\tau))$ is not parabolic. Thus $B(\tau) \in \mathcal{S}(\Gamma, k+2, \rho, v)$.

Therefore the map

$$\widetilde{\mu} : \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v}) \oplus \mathcal{S}(\Gamma, k+2, \rho, v) \to \widetilde{H}^1_{v,\rho,p}(\Gamma, P_k). \tag{3.115}$$

is onto. The proof of Theorem 3.2 is complete.

3.6 End of proof of Theorem 3.3

Now we want to show that the map μ of theorem 3.3 is onto. We have already seen that

$$\widetilde{\mu} : \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v}) \oplus \mathcal{S}(\Gamma, k+2, \rho, v) \to \widetilde{H}^1_{v,\rho,p}(\Gamma, P_k). \tag{3.116}$$

is onto. Let $\{Q_V(\tau) : V \in \Gamma\}$ be a cocycle in a nonparabolic class, with the polynomial corresponding to the translation given by

$$Q^{(j)}_T(\tau) = b^{(j)}_k \tau^k + b^{(j)}_{k-1}\tau^{k-1} + \ldots + b^{(j)}_1 \tau + b^{(j)}_0. \tag{3.117}$$

We will construct a vector-valued modular form $l(\tau) \in \mathcal{M}(\Gamma, k+2, \rho, v)$, such that the cohomology class of the $(k+1)$-fold integral, $L(\tau)$, will give rise to a cocycle $\{p_V(\tau) : V \in \Gamma\}$, where

$$p_T(\tau) = Q_T(\tau). \tag{3.118}$$

Let

$$l(\tau) = -k! \sum_{i=1}^{p} b^{(j)}_k P(\tau, \rho, k+2, v, 0, \Gamma, j), \tag{3.119}$$

where $b^{(i)}_k$ is given by (3.117), and $P(\tau, \rho, k+2, v, 0, \Gamma, i)$ by (3.13). Now the $(k+1)$-fold integral of $l(\tau)$ is given by

$$L^{(j)}(\tau) = -k! b^{(j)} \sum_{m \geq m_j} c^{(j)}_m \left(2\pi i(m+m_j)\right)^{-k-1} e^{2\pi i(m+m_j)\tau} + \delta_j \tau^{k+1} + p^{(j)}(\tau), \tag{3.120}$$

where c_m are the Fourier coefficients of $P(\tau,\rho,k+2,v,0,\Gamma,j)$, $p^{(j)}(\tau)$ is a polynomial of degree at most k, whose coefficients can be chosen at our convenience and δ_j is given by

$$\delta_j = \begin{cases} -\frac{b^{(j)}}{(k+1)}, & m_j = 0 \\ 0, & m_j \neq 0 \end{cases} . \tag{3.121}$$

Now we choose the coefficients of $p^{(j)}(\tau)$:

$$p^{(j)}(\tau) = a_k^{(j)}\tau^k + a_{k-1}^{(j)}\tau^{k-1} + \ldots + a_1^{(j)}\tau + a_0^{(j)}. \tag{3.122}$$

If $m_j \neq 0$, we want to choose the coefficients so that

$$p^{(j)}(\tau) - (p(\tau) \mid_{-k,v,\rho} T(\tau))^{(j)} = Q_T^{(j)}(\tau). \tag{3.123}$$

Note that

$$\begin{aligned} p^{(j)}(\tau) - (p(\tau) \mid_{-k,v,\rho} T(\tau))^{(j)} &= p^{(j)}(\tau) - e^{-2\pi i m_j} p^{(j)}(\tau+1) \\ &= p^{(j)}(\tau) - e^{-2\pi i m_j} \sum_{h=0}^{k}\sum_{s=0}^{h} \begin{pmatrix} h \\ s \end{pmatrix} a_h^{(j)}\tau^{h-s}. \end{aligned} \tag{3.124}$$

Therefore we can set up the following linear system:

$$\begin{pmatrix} 1-e^{-2\pi i m_j} & d^{(1,2)} & d^{(1,3)} & \ldots & d^{(1,k+1)} \\ 0 & d^{(2,2)} & d^{(2,3)} & \ldots & d^{(2,k+1)} \\ 0 & 0 & d^{(3,3)} & \ldots & d^{(3,k+1)} \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1-e^{-2\pi i m_j} \end{pmatrix} \begin{pmatrix} a_0^{(j)} \\ a_1^{(j)} \\ \vdots \\ a_{k-1}^{(j)} \\ a_k^{(j)} \end{pmatrix} = \begin{pmatrix} b_0^{(j)} \\ b_1^{(j)} \\ \vdots \\ b_{k-1}^{(j)} \\ b_k^{(j)} \end{pmatrix}, \tag{3.125}$$

where $d^{(h,l)}$ is given by

$$d^{(h,l)} = \begin{cases} 1-e^{-2\pi i m_j}\begin{pmatrix} l-1 \\ l-h \end{pmatrix} & h = l \\ -e^{-2\pi i m_j}\begin{pmatrix} l-1 \\ l-h \end{pmatrix} & h \neq l. \end{cases} \tag{3.126}$$

This linear system has a unique solution for the $a^{(j)}$'s, and satisfies (3.123).

If $m_j = 0$, we want to choose the coefficients so that

$$\delta_j \tau^{k+1} + p^{(j)}(\tau) - \left(\delta_j \tau^{k+1} + p(\tau) \mid_{-k,v,\rho} T(\tau)\right)^{(j)} = Q_T^{(j)}(\tau). \tag{3.127}$$

Note that

$$\delta_j \tau^{k+1} - \left(\delta_j \tau^{k+1} \mid_{-k,v,\rho} T(\tau)\right)^{(j)} = -\delta_j \sum_{s=0}^{k} \begin{pmatrix} k+1 \\ s+1 \end{pmatrix} \tau^{k-s}. \tag{3.128}$$

Now we can set up the system of equations

$$\begin{pmatrix} 0 & d^{(1,2)} & d^{(1,3)} & \dots & d^{(1,k+1)} \\ 0 & 0 & d^{(2,3)} & \dots & d^{(2,k+1)} \\ 0 & 0 & 0 & \dots & d^{(3,k+1)} \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_0^{(j)} \\ a_1^{(j)} \\ \vdots \\ a_{k-1}^{(j)} \\ a_k^{(j)} \end{pmatrix} = \begin{pmatrix} b_0^{(j)} + \delta_j \\ b_1^{(j)} + \delta_j \begin{pmatrix} k+1 \\ k \end{pmatrix} \\ \vdots \\ b_{k-1}^{(j)} + \delta_j \begin{pmatrix} k+1 \\ 2 \end{pmatrix} \\ b_k^{(j)} + \delta_j (k+1) \end{pmatrix}, \tag{3.129}$$

where

$$d^{(h,l)} = -\begin{pmatrix} l-1 \\ l-h \end{pmatrix}. \tag{3.130}$$

This system of equations has a solution for the $a^{(j)}$'s, since by (3.121) we see that

$$b_k^{(j)} + \delta_j (k+1) = 0. \tag{3.131}$$

And by the choice of the coefficients we have that (3.127) holds. Note that this solution is unique except for $a_0^{(j)}$, which can be chosen to be whatever we want since $m_j = 0$. Sumarizing, we have found a vector-valued modular form $l(\tau) \in \mathcal{M}(\Gamma, k+2, \rho, v)$, such that the cohomology class, $\beta(l)$, of the $(k+1)$-fold integral, $L(\tau)$, has the same cocycle for T as the given nonparabolic cohomology class. Therefore we can say that a given nonparabolic cohomology class can be written as

$$\beta(l) + \widetilde{p}, \tag{3.132}$$

where $\widetilde{p}$ is a parabolic cohomology class in $\widetilde{H}^1_{v,\rho,p}(\Gamma, P_k)$. We saw in Theorem 3.2 that

$$\mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v}) \oplus \mathcal{S}(\Gamma, k+2, \rho, v) \cong \widetilde{H}^1_{v,\rho,p}(\Gamma, P_k); \tag{3.133}$$

therefore, there exists $g \in \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v})$ and $h \in \mathcal{S}(\Gamma, k+2, \rho, v)$ such that

$$\widetilde{p} = \alpha(g) + \beta(h). \tag{3.134}$$

Hence the given nonparabolic cohomology class in $H^1_{v,\rho,p}(\Gamma, P_k)$ is

$$\alpha(g) + \beta(h) + \beta(l), \tag{3.135}$$

or, what is the same,

$$\alpha(g) + \beta(h+l), \tag{3.136}$$

where $g \in \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v})$ and $h + l \in \mathcal{M}(\Gamma, k+2, \rho, v)$. Therefore the map

$$\mu : \mathcal{S}(\Gamma, k+2, \widehat{\rho}, \widehat{v}) \oplus \mathcal{M}(\Gamma, k+2, \rho, v) \to H^1_{v,\rho,p}(\Gamma, P_k) \tag{3.137}$$

is onto and the proof is complete.

REFERENCES

[1] T.M. Apostol, Modular Functions and Dirichlet Series in Number Theory, Second Edition, Springer-Verlag, new york-Heidelberg-Berlin, 1990.

[2] J. Dixon, The number of steps in the Euclidean algorithm, J. Number Theory 2 (1970), 414-422.

[3] M. Eichler, Grenzkreisgruppen und kettenbruchartige Algorithmen, Acta Arith. 11 (1965), 169-180.

[4] S. Y. Husseini and M. I. Knopp, Eichler cohomology and automorphic forms, Illinois J. Math. 15 (1971), 565-577.

[5] M. Knopp, Automorphic forms of nonnegative dimension and exponential sums, Michigan Math. J. 7 (1960), 257-287.

[6] M. Knopp, Construction of automorphic forms on H-groups and supplementary Fourier series, Trans. Amer. Math. Soc. 103 (1962), 168-188.

[7] M. Knopp, Correction to "Construction of automorphic forms on H-groups and supplementary Fourier series", Trans. Amer. Math. Soc. 106 (1963), 341-345.

[8] M. Knopp, Modular Functions in Analytic Number Theory, Markham Publishing, Illinois, 1970.

[9] M. Knopp, Some new results on the Eichler cohomology of automorphic forms, Bull. Amer. Math. Soc. 80 (1974), 607-632.

[10] M. Knopp and G. Mason, Generalized modular forms, J. Number Theory 99 (2003), 1-28.

[11] M. Knopp and G. Mason, On vector-valued modular forms and their Fourier coefficients, Acta Arith. 110 (2003), no. 2, 117-124.

[12] M. Knopp and G. Mason, Vector-valued modular forms and Poincare series, Illinois J. of Math. 48 (2004), 1345-1366.

[13] J. Lehner, Cohomology of vector-valued automorphic forms, Math. Ann. 204 (1973), 155-176

[14] H. Rademacher, On the partition function $p(n)$, Proc. London Math. Soc. 43 (1937), 241-254.

[15] H. Rademacher, A convergent series for the partition function $p(n)$, Proc. Nat. Ac. Sci USA 23 (1937), 78-84.

[16] H. Rademacher, The Fourier coefficients of the modular invariant $J(\tau)$, Am. J. Math. 60 (1938), 501-512.

[17] H. Rademacher, The Fourier series and the functional equation of the absolute modular invariant $J(\tau)$, Am. J. Math. 61 (1939), 237-248.

[18] H. Rademacher, On the expansion of the partition function in a series, Ann. Math. 44 (1943), 416-422.

[19] H. Rademacher and H. Zuckerman, On the the Fourier coefficients of certain modular forms of positive dimension, Ann. Math. 39 (1938), 433-462.

[20] G. Watson, A Treatise on the Theory of Bessel Functions, Second Edition. Cambridge University Press, 1944