THE CENTER OF $U(\mathfrak{sl}_2)$ AND THE BASICS OF ITS QUANTUM ANALOG

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I intend this to merely be an exposition of several well known results. I will loosely follow the treatments of Lang and Kassel. Let us begin with a few basics of $U(\mathfrak{g})$.

Given any Lie algebra $\mathfrak{g}$ we define its *universal enveloping algebra* $U(\mathfrak{g})$ to be the quotient of the free algebra

$$F(\mathfrak{g}) = \{\text{free algebra generatedby X} \in \mathfrak{g}\}$$

by the relations given by the bracket on $\mathfrak{g}$

$$U(\mathfrak{g}) = \frac{F(\mathfrak{g})}{(XY - YX - [X,Y])}$$

In the case of $\mathfrak{sl}_2$ (here we mean $\mathfrak{sl}_2(\mathbb{C})$, but the same holds true for $\mathfrak{sl}_2(\mathbb{R})$.) We have a basis:

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are subject to:

$$[H, X_+] = 2X_+, \quad [H, X_-] = -2X_-, \quad [X_+, X_-] = H$$

Then $U(\mathfrak{sl}_2) = \frac{F(X_+, X_-, H)}{\text{relations}}$.

Theorem 1. *(Poincare-Birkhoff-Witt)*
$\{X_+^p H^q X_-^r\}$ *for* $p, q, r \geq 0$ *integers form a basis for* $U(\mathfrak{sl}_2)$.

This theorem can be found in many books on Lie algebras so I won't prove it here. However Lang uses an interesting approach to this theorem so let us at least mention it. In Lang's text he proves that the map $U(\mathfrak{sl}_2) \longrightarrow Der(C^\infty(SL_2))$ is injective by noting that $\delta_a(x) = [a, x]$ is a derivation.
We shall also use the fact that δ_a is a derivative $\forall a$ when we get to the Casimir element.
Note: $\delta_a(xy) = x\delta_a(y) + \delta_a(x)y$.
Furthermore

$$\delta_a(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} x_1 \ldots \delta_a(x_j) \ldots x_n$$

Now that we have some grasp on the basic structure of

$$U(\mathfrak{sl}_2) = \sum_{p,q,r \geq 0} c_{pqr} X_+^p H^q X_-^r$$

Theorem 2. *The centralizer of* H *is the set of all monomials* $X_+^p H^q X_-^p$

Proof. Consider $[H, X_+^p H^q X_-^r] = 0$. Then the claim is that $p = r$. Using the standard representation for $\mathfrak{sl}_2$ we have the following relations:

$$\begin{array}{ll} X_+^p H^q = (H-2p)^q X_+^p & X_-^p H^q = (H+2p)^q X_-^p \\ H^q X_+^p = X_+^p (H+2p)^q & H^q X_-^p = X_-^p (H-2p)^q \end{array}$$

Then

$$\begin{aligned} [H, X_+^p H^q X_-^r] &= H X_+^p H^q X_-^r - X_+^p H^q X_-^r H \\ &= X_+^p (H+2p) H^q Y^r - X_+^p H^q (H+2r) Y^r \\ &= X^p H^q [2p-2r] Y^r \\ &= 0 \end{aligned}$$

Which is true $\Leftrightarrow p = r$. Thus we conclude:

$$[H, \sum_{p,q,r \geq 0} c_{pqr} X_+^p H^q X_-^r] = 0 \Leftrightarrow p = r$$

for all mononomials by linearity of the bracket. □

Certainly this is a start since $Z(H) \supseteq Z(U(\mathfrak{sl}_2))$. We could attempt to compute the centralizers for $X_\pm$, but these do not have such a nice form. Furthermore we would only know

$$Z(U(\mathfrak{sl}_2)) \subseteq Z(H) \cap Z(X_+) \cap Z(X_-)$$

we won't necessarily have equality a priori.

Let us now define a unique linear multiplicative homomorphism $h : Z(H) \longrightarrow \mathbb{C}(H)$ given by:

$$h(X_+^p H^q X_-^p) = \begin{cases} H^q & p = 0 \\ 0 & p > 0 \end{cases}$$

We'll show h is multiplicative and injective on $Z(H)$. $h(Y)$, the image of $Y \in Z(H)$ is characterized by congruence $Y \equiv h(Y) \mod UX_-$. Then letting $Y_1 \equiv h(Y_1) \mod UX_-$ and $Y_2 \cong h(Y_2) \mod UX_-$, we have:

$$Y_2 Y_1 \cong h(Y_2) h(Y_1) \mod UX_- H + UX_-$$

Using the commutation relations between X_- and H and that X_- is an eigenvector of $ad(H)$ we get that h is multiplicative. To show injectivity we need $h(Y) = 0 \Rightarrow Y = 0$.

Let $Y = \sum_q c_q X_+^r H^q X_-^r + \sum_{p>r} c_{pq} X_+^p H^q X_-^p$, for $r \geq 1$. We show Y can't commute with X_-. This amounts to picking an irreducible representation with lowest weight λ. Let us denote a vector of weight λ by v_λ. We know for any irreducible representation V we have the isomorphism $V \simeq \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ and in the case of $\mathfrak{sl}_2$ we have $X_\pm : V_\lambda \longrightarrow V_{\lambda \pm 2}$. We only need to pick a vector $v_{\lambda+2r}$ so that for $p > r$ $X_+^p H^q X_-^p$ annihilates $v_{\lambda+2r}$ and $X_- v_{\lambda+2r}$. Then the effect of terms involving r^{th} powers of $X_\pm$ has the same effect as Y. i.e.

$$(\sum_q c_q X_+^r H^q X_-^r + \sum_{p>r} X_+^p H^q X_-^p) v_{\lambda+2r} = (\sum c_q X_+^r H^q X_-^r) v_{\lambda+2r}$$

Then we have $Y v_{\lambda+2r} = \mu(r) \sum c_q (\lambda)^q v_{\lambda+2r}$ for $\mu(r) \in \mathbb{C}^*$. Applying X yields a nonzero vector for appropriate λ. However

$$Y E_- v_{\lambda+2r} = 0 \neq E_- Y v_{\lambda+2r}, \quad [Y, E_-] \neq 0$$

Then if $Y \in Z(U)$ and $h(Y) = 0 \Rightarrow Y \equiv 0$.

Definition 3. The *Casimir Operator* is $C = \frac{1}{2}H^2 + (X_+X_- + X_-X_+)$.

We'll show explicitly that $[H, C] = [X_+, C] = [X_-, C]$.

(1)

$$\begin{aligned} [H,C] &= \tfrac{1}{2}[H,H^2] + [H, X_+X_-] + [X, X_-X_+] \\ &= 0 + (X_+[H,X_-] + [H,X_+]X_-) + (X_-[H,X_+] + [H,X_-]X_+) \\ &= -2X_+X_- + 2X_+X_- - 2X_-X_+ + 2X_-X_+ \\ &= 0 \end{aligned}$$

(2)

$$\begin{aligned} [X_+,C] &= \tfrac{1}{2}[X_+,H^2] + [X_+, X_+X_-] + [X_+, X_-X_+] \\ &= \tfrac{1}{2}(H[X_+,H] + [X_+,H]H) + (X_+[X_+X_-] + 0) + (0 + [X_+,X_-]X_+) \\ &= -HX_+ - X_+H + X_+H + HX_+ \\ &= 0 \end{aligned}$$

(3) $[X_-, C] = 0$ is the same as (2) above.

Theorem 4. *$Z(U(\mathfrak{sl}_2)) \simeq \mathbb{C}[t]$ under the homomorphism $C \mapsto t$. This is know as the* Harish-Chandra homomorphism*.*

The idea of the proof is to use that h is injective. Also $h(C) = H^2$ and we show that only even powers of H occur in the center of $U(\mathfrak{sl}_2)$ under the image of h. We proceed as before this time letting $Y = \sum_q c_qH^q + \sum_{p\geq 1} X_+^pH^qX_-^p$ and showing $Yv_\lambda = P(\lambda - 1)v_\lambda$ for some polynomial P and $P(-\lambda + 1) = P(\lambda - 1)$ hence P is even.
One final remark about $U(\mathfrak{sl}_2)$. At this juncture we can explicitly give $Z(H)$. We know $Z(H) \supset \mathbb{C}[H]$ and $Z(H) \supset Z(U(\mathfrak{sl}_2)) = \mathbb{C}[t]$ where t represents the Casimir operator. It turns out that $Z(H) \cong \mathbb{C}[t, H]$. The proof of this statement requires checking some more commutation relations and induction then appealing to the fact that the centralizer of $H = \{\sum_{p,q} c_{pq}X_+^pH^qX_-^p\}$.

The Quantum Enveloping Algebra of $\mathfrak{sl}_2$

There are two somewhat standard presentations, but we'll use the one which more closely resembles $U(\mathfrak{sl}_2)$ which is the special case if we allow $q = 1$. With the exception of showing the relation between $U(\mathfrak{sl}_2)$ and $U_1(\mathfrak{sl}_2)$ we will consider $q \in \mathbb{C}^*$ not a root of unity.

Definition 5. The algebra $U_q(\mathfrak{sl}_2)$ is generated by 5 elements: E, F, K, K^{-1}, L with the following relations:

$$\begin{aligned} KK^{-1} = K^{-1}K &= 1 & KEK^{-1} &= q^2E \\ KFK^{-1} &= q^{-2}F & [E,F] &= L \\ (q-q^{-1})L &= K - K^{-1} & [L,E] &= q(EK + K^{-1}E) \end{aligned}$$

and $[L, F] = -q^{-1}(FK + K^{-1}F)$

We see immediately that if $q = 1$ K is central and

$$K^2 = 1, \quad E = X_+, \quad F = X_-, \quad L = H$$

So it might behoove us to examine $U_q(\mathfrak{sl}_2)$ more closely, but that's just a suggestion. It should be noted that the representations of $U_q(\mathfrak{sl}_2)$ have a very familiar feel in light of $U(\mathfrak{sl}_2)$. Let us now attempt to describe $Z_q := Z(U_q(\mathfrak{sl}_2))$. Again we exhibit the Casimir operator:

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} \quad also \quad C_q = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

To check this we need only check $[E, C_q] = [K, C_q] = [F, C_q] = 0$ and then appealing to our defining relations.

1. Since $KEFK^{-1} = EF$ we get $[K, C_q] = 0$
2. $$\begin{aligned} EC_q &= EFE + E(\frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}) \\ &= EFE + (\frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2})E \\ &= C_qE \end{aligned}$$
3. F is done similarly

Definition 6. U_q^K is the subalgebra of $U_q(\mathfrak{sl}_2)$ such that $[K, X] = 0, \ \forall X \in U_q^K$.

Lemma 7.

$$X \in U_q^K \Leftrightarrow X = \sum_{i \geq 0} F^i P_i E^i$$

where $P_i \in \mathbb{C}[K, K^{-1}]$.

Proof. First $\{F^i K^l E^j\}_{i,j \geq 0, \ l \in \mathbb{Z}}$ is a basis for $U_q(\mathfrak{sl}_2)$. Now $K(F^i K^l E^i)K^{-1} = q^{2(j-i)} F^i K^l E^i \Rightarrow$ commutes when $i = j$. □

As before let us define a homomorphism

$$\phi : U_q^K : Z(K) \longrightarrow \mathbb{C}[K, K^{-1}]$$

by $\phi(F^i P_i E^i) = \begin{cases} P_i & i = 0 \\ 0 & i \neq 0 \end{cases}$

This ϕ is the *Harish-Chandra* homomorphism.

Much of the theory follows as before with similar arguments. From here I'll state some results and leave out most of the proofs. For full details see Kassel pp. 130-134.

The Harish-Chandra homomorphism allows us to express the action of Z_q on the highest weight vector.

Proposition 8. *Let V_λ be a highest weight $U_q(\mathfrak{sl}_2)$ module with weight λ. Then $\forall z \in Z_q$ and $v_\lambda \in V_\lambda$ we have $zv_\lambda = \phi(z)(\lambda)v_\lambda$ where $\phi(z)$ is a Laurent polynomial in K with $\phi(z)(\lambda)$ its value at λ.*

Example 9.

$$\phi(C_q) = \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

hence C_q acts on v_λ by $C_q v_\lambda = \frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2} v_\lambda$.

Lemma 10. *$\phi\,|_{Z_q}$ is injective i.e. if $z \in Z_q$ then $\phi(z) = 0 \Rightarrow z = 0$*

One last bit of notation let $\tilde{P}(\lambda) \doteq P(q^{-1}\lambda)$

Lemma 11. *$\forall z \in Z_q$ we have $\tilde{\phi(z)}(\lambda) = \tilde{\phi(z)}(\lambda^{-1})$, providing us a beautiful symmetry condition.*

Example 12. $\tilde{\phi(C_q)}(\lambda) = \tilde{\phi(C_q)}(\lambda^{-1})$ as:

$$C_q = EF + \frac{q^{-1}K + q\ K^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

Lemma 13. *Any $P \in \mathbb{C}[K, K^{-1}]$ satisfying $P(\lambda) = P(\lambda^{-1})$ is a polynomial in $K + K^{-1}$. Proof is by induction.*

Theorem 14. *$Z_q \simeq \mathbb{C}[t_q]$ where t_q represents C_q. The restriction of ϕ to Z_q is an isomorphism onto the subalgebra $\mathbb{C}[qK + q^{-1}K^{-1}] \subseteq \mathbb{C}[K, K^{-1}]$.*

Proof. We know $\phi\,|_{Z_q}$ is injective. By our previous two lemmas its image is generated by $qK + q^{-1}K^{-1}$ inside $\mathbb{C}[K, K^{-1}]$. Consider C_q, we know $\phi(C_q) = \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$ which shows $\phi(Z_q)$ is the entire subalgebra and C_q generates the center. $\square$

The general idea for considering $U_q(\mathfrak{g})$ when $q^n \neq 1\ \forall n$ is that much of the theory is the same as $U(\mathfrak{g})$ we only need to be careful in carrying around our $q's$.

REFERENCES

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