PREPARATION FOR ORALS

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Contents

These notes are in no way intended to be comprehensive or complete. They are intended to be notes for my study in preparation for my Oral Exam at Northwestern University. Furthermore these notes are not my original work. I will try to fill in as many details as possible and give as many examples as I can work through.

1. Pseudodifferential Operator Basics

An extensive amount of literature on this material is available. My presentation will be loosely based on Melrose and Shubin. I'll only present examples relevant to me and notes of interest.

1.1. Symbols.

The first important thing to point out is that there are several standard notations for symbol spaces. Melrose likes S^m_∞ while Shubin uses $S^m_{\rho,\delta}$. My notation will simply be S^m and by this I will mean the same thing as Melrose's S^m_∞ and Shubin's $S^m_{1,0}$. I will start with a more general definition.

Definition 1. The space $S^m_{\rho,\delta}(\mathbb{R}^n, \mathbb{R}^p)$ of symbols of order m is the set of functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^p)$ that satisfy the estimates

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \leq C(1+|\xi|)^{m-\rho|\beta|-\delta|\alpha|} \tag{1.1.1}$$

where $1-\rho \leq \delta \leq \rho \leq 1$ and α and β are multiindices and $C \in \mathbb{R}$.

In my case I will take $\rho = 1$ and $\delta = 0$ so that the condition becomes

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \leq C(1+|\xi|)^{m-|\beta|} \tag{1.1.2}$$

A further simplification is that I will usually take $n = p$ unless otherwise noted so the notation becomes $S^m(\mathbb{R}^n)$ or S^m instead of $S^m_{1,0}(\mathbb{R}^n, \mathbb{R}^n)$.

What this symbol estimate is saying is that we wish to bound the growth of these functions in the ξ variable. Melrose's notation of S^m_∞ refers to the assumption that these functions have uniform boundedness in the x variable. A theoretically useful result is that S^m is a metrizable topological space. We can define seminorms as follows

$$\|a\|_{N,m} = \sup_{x\in\mathbb{R}^n, \xi\in\mathbb{R}^p} \max_{|\alpha|+|\beta|\leq N} (1+|\xi|)^{-m+|\beta|} |D^\alpha_x D^\beta_\xi a(x,\xi)| < \infty \tag{1.1.3}$$

This says that we can take the best possible C in the estimates above. Given these norms we can define a metric as follows

$$d(a,b) = \sum_{N\geq 0} 2^{-N} \frac{\|a-b\|_{N,m}}{1+\|a-b\|_{N,m}} a, b \in S^m(\mathbb{R}^n, \mathbb{R}^p).$$

We can essentially put this metric out of mind. It is not a practical tool with which to make calculations, but it is useful to know it exists. Now let us list a few basic properties of symbols. There is an obvious inclusion

$$S^m(\mathbb{R}^n) \hookrightarrow S^{m'}(\mathbb{R}^n) \qquad \forall m' \geq m$$

which we can see by simply observing

$$(1+|\xi|)^m \leq C(1+|\xi|)^{m'} \qquad \forall \xi \in \mathbb{R}^n \iff m' \geq m$$

Now we define the spaces $S^{\infty}(\mathbb{R}^n)$ and $S^{-\infty}(\mathbb{R}^n)$ to be the union and respectively intersection of all $S^m(\mathbb{R}^n)$.

$$S^{\infty}(\mathbb{R}^n) = \bigcup_m S^m(\mathbb{R}^n) \tag{1.1.4}$$

$$S^{-\infty}(\mathbb{R}^n) = \bigcap_m S^m(\mathbb{R}^n) \tag{1.1.5}$$

The last two results for this section will be important in later considerations.

Lemma 2. *For any $m \in \mathbb{R}$ $S^{-\infty}(\mathbb{R}^n)$ is dense in $S^m(\mathbb{R}^n)$ in the topology of $S^{m'}(\mathbb{R}^n)$ for any $m' > m$.*

Lemma 3. $S^m(\mathbb{R}^n) \cdot S^{m'}(\mathbb{R}^n) \subset S^{m+m'}(\mathbb{R}^n)$

Both of these lemmas can be proved using the metric topology defined above. Now let us move onto the real goal of this section that is to define pseudodifferential operators.

1.2. **ΨDO on $\mathbb{R}^n$.** In order to properly define ΨDO we should first recall some basic properties of the Fourier transform.

Definition 4. The *Fourier transform* of $\phi \in \mathcal{S}(\mathbb{R}^n)$ (A Schwartz function) denoted $\mathcal{F}\phi$ or $\hat{f}$ is defined to be

$$\mathcal{F}\phi(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\phi(x)dx$$

Now let us see two very important properties of $\mathcal{F}$

$$\mathcal{F}(D_j\phi) = \xi_j\mathcal{F}\phi = \xi_j\hat{\phi} \tag{1.2.1}$$

$$\mathcal{F}(x_j\phi) = -D_{\xi_j}\mathcal{F}\phi \tag{1.2.2}$$

Here $D_j = -id/dx^j$ we take the scalar multiple of d/dx so that it is a self adjoint operator.

Remark 5. Notice how convenient it is to now define the derivative as an integral operator. For example on $\mathbb{R}$

$$Du = (2\pi)^{-1}\int_{\mathbb{R}_y\times\mathbb{R}_\xi} e^{i(x-y)\xi}\xi u(y)dyd\xi = (2\pi)^{-1}\int_{\mathbb{R}} e^{ix\xi}\xi\hat{u}(\xi)d\xi$$

More generally we can naively take a polynomial differential operator $P(x, D)$ on $\mathbb{R}^n$ and write it as an integral

$$P(x,D)u = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}P(x,\xi)u(y)dyd\xi = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}P(x,\xi)\hat{u}(\xi)d\xi$$

Definition 6. A *Pseudodifferential Operator* of order m on $\mathbb{R}^n$ is an integral operator of the form

$$Au(x) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi$$

where $a(x, y, \xi) \in (1 - |x - y|^2)^{w/2}S^m(\mathbb{R}^{2n} \times \mathbb{R}^n)$ for some $w \in \mathbb{R}$.

Notice here the importance of the extra somewhat unexpected term $(1-|x-y|^2)^{w/2}$. This is to account for growth away from the diagonal $x = y$. It will play an important role when we discuss reduction, composition, and quantization.

1.3. **Reduction, Composition, and Asymptotic Summation.** Here I'll present a menagerie of things which are reasonably short and all relatively important. The first is reduction.

Theorem 7. *Any* $A \in \Psi^m(\mathbb{R}^n)$ *can be written uniquely in terms of a symbol* $a(x,\xi) \in S^m(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$ *as opposed to* $a(x,y,\xi) \in (1-|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi)$

Proposition 8. *Let*

$$(1-|x-y|)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n) \ni a \mapsto I(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi$$

Then $I(a) \in (1+|x|^2+|y|^2)^{w/2}C^0(\mathbb{R}^{2n})$ *provided* $m < -n$ *(for absolute convergence). Furthermore the range of* I *(for any* w*) is the same as* $I|_{i(a)}$ *where* $i : S^m(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ *is the inclusion* $i(a(x,\xi)) = a(x,\xi)$.

Melrose gives a more explicit result for the first part. He further says that I extends by continuity to
$I : (1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$ for each $w, m \in \mathbb{R}$ in the topology of m' where $m' > m$. This result is proved using the triangle inequality in the guise of

$$(1+|x-y|) \leq (1+|x|)(1+|y|).$$

And integration by parts in the guises of

$$(1+\xi\cdot D_x)e^{i(x-y)\cdot\xi} = (1+|\xi|^2)e^{i(x-y)\cdot\xi}$$
$$(1-\xi\cdot D_y)e^{i(x-y)\cdot\xi} = (1+|\xi|^2)e^{i(x-y)\cdot\xi}$$

Now to prove the proposition. I will pull this mostly from Melrose's notes (§2.4) and attempt to fill in details.

Proof. Suppose $a \in (1+|x-y|^2)^{w/2}S^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ for some w, then using integration by parts again we get

$$I((x_j-y_j)a) = I(-D_{\xi_j}a); j = 1,\ldots,n \tag{1.3.1}$$

Something slightly tricky that one should notice is the fact that if $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ then $D_{\xi_j}a \in S^{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ which is to say that an operator with amplitude $(x_j - y_j)a(x,y,\xi)$ has order $m-1$ despite the appearance that it is of order m.

In order to show we can reduce the symbol from $a(x,y,\xi)$ to $a(x,\xi)$ we exploit another of our favorite tools from calculus known as Taylor's theorem, and expand around $x = y$.

$$a(x,y,\xi) = \sum_{|\alpha|\leq N-1}\frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha(D_y^\alpha a)(x,x,\xi) + \sum_{|\alpha|\leq N-1}\frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^\alpha \cdot R_{N,\alpha}(x,y,\xi) \tag{1.3.2}$$

Where $R_{N,\alpha}$ is the remainder term of order N given by

$$R_{N,\alpha}(x,y,\xi) = \int_0^1 (1-t)^{N-1}(D_y^\alpha a)(x,(1-t)x+ty,\xi)dt$$

Another thing to notice is that multiplication by $(x-y)^\alpha$ changes the exponent in our estimate.

$$(x-y)^\alpha(D_y^\alpha a)(x,y,\xi) \in (1+|x-y|^2)^{\frac{w+|\alpha|}{2}}S^m(\mathbb{R}^{2n};\mathbb{R}^2).$$

The final trick is to apply (1.3.1) iteratively to pull down the order of $a(x,y,\xi)$. Then if $A\Psi^m(\mathbb{R}^n)$ with kernel $I(a)$ we can write A as follows

$$A = \sum_{j=0}^{N-1} A_j + R_N, \quad A_j \in \Psi^{m-j}(\mathbb{R}^n), \quad R_N \in \Psi^{m-N}(\mathbb{R}^n) \tag{1.3.3}$$

That gives us that A_j has kernel

$$I(\sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!}(D_y^\alpha D_\xi^\alpha)(x,x,\xi)) \tag{1.3.4}$$

This leads us to an apparently messy trap. Can we sum this series? Not exactly, but this is where we come to asymptotic summation. ☐

Definition 9. We say $a \in S^m(\mathbb{R}^p;\mathbb{R}^n)$ is *asymptotically summable* if there exists a sequence $a_j \in S^{m-j}(\mathbb{R}^p;\mathbb{R}^n)$ such that

$$\forall N, \quad a - \sum_{j=0}^{N-1} a_j \in S^{m-N}(\mathbb{R}^p;\mathbb{R}^n) \tag{1.3.5}$$

We write this relation

$$a \sim \sum_{j=0}^{\infty} a_j$$

Proposition 10. *Any series $a_j \in S^{m-j}(\mathbb{R}^p;\mathbb{R}^n)$ is asymptotically summable and the summation is unique up to an additive term in $S^{-\infty}(\mathbb{R}^p;\mathbb{R}^n)$.*

Proof. Existence involves some cut off functions and a construction due to Emile Borel. Uniqueness is easier. Suppose we have two symbols $a \sim \sum_{j=0}^{\infty} a_j$ and $a' \sim \sum_{j=0}^{\infty} a_j$. Then taking the difference we get

$$a - a' = (a - \sum_{j=0}^{N-1} a_j) - (a' - \sum_{j=0}^{N-1} a_j) \in S^{m-N}(\mathbb{R}^p;\mathbb{R}^n)$$

but this happens for all N hence $a - a' \in S^{-\infty}(\mathbb{R}^p;\mathbb{R}^n)$. ☐

Let us now apply our last few results to find $b \in S^m(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$ satisfying

$$b(x,\xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!}(D_y^\alpha)(x,x,\xi).$$

Let B be the operator with kernel $I(b)$. Then applying reduction and asymptotic summation we get

$$A - B = \sum_{j=0}^{N-1} A_j + R_N - B$$

$$B = \sum_{j=0}^{N-1} A_j + R'_N, \quad R'_N \in \Psi^{m-N}(\mathbb{R}^n)$$

$$A - B \in \Psi^{-\infty}(\mathbb{R}^n) = \bigcap_N \Psi^N(\mathbb{R}^n)$$

We've come to another annoying subtlety. We don't know that $A-B$ has kernel $I(c)$ with $c \in S^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$. All we know at this point is that $A - B$ has kernel

$I(c_N)$ with $c_N \in S^{m-N}(\mathbb{R}^{2n};\mathbb{R}^n)$ for each N. This leads us to a wonderful new statement.

Proposition 11. *An operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is an element of $\Psi^{-\infty}(\mathbb{R}^n)$ $\iff$ its Schwartz kernel is C^∞ and satisfies the estimates*

$$|D_x^\alpha D_y^\beta K(x,y)| \leq C(1+|x-y|)^{-N} \;\; \forall \alpha,\beta,N \tag{1.3.6}$$

The proof is not particularly interesting or enlightening. It uses our old friend integration by parts again. Let me quickly mention that if $A \in \Psi^{-\infty}$ then A is called *smoothing operator*. The reason for this is obviously that for any function $u \in C^k$ the function $Au \in C^\infty$. The proof of this fact is easy enough when we write

$$Au(x) = \int K_A(x,y)u(y)dy$$

Now we require $K_A(x,y) \in C^\infty$ and satisfying some fairly strict estimates. What happens then when we take derivatives in the x variable?

$$\frac{d}{dx}Au(x) = \frac{d}{dx}\int K_A(x,y)u(y)dy = \int \frac{\partial}{\partial x}K_A(x,y)u(y)dy$$

but of course $K_A \in C^\infty$ so we can take as many derivatives in x as we'd like. The only issue then is convergence, but we get help from our strict estimates.

Now we are ready to state the composition theorem.

Theorem 12. *The space Ψ^∞ is an order filtered $*$-algebra on $\mathcal{S}(\mathbb{R}^n)$. In particular we wish to show*

$$A \in \Psi^m(\mathbb{R}^n) \implies A^* \in \Psi^m(\mathbb{R}^n) \tag{1.3.7}$$

$$A \in \Psi^m(\mathbb{R}^n),\; B \in \Psi^{m'}(\mathbb{R}^n) \implies A \circ B \in \Psi^{m+m'}(\mathbb{R}^n) \tag{1.3.8}$$

We again have a small annoyance to deal with in that Ψ^∞ is filtered and not graded. Worse things could happen. At least we have a somewhat natural filtration for a spectral sequence which will be used in calculating the Hochschild Homology. Nonetheless we can easily see that Ψ^∞ is not graded by simply observing a change of variables.

Example 13. This example illustrates the fact that Ψ^∞ is not graded. The difficulty here will later lead us to the principal symbol.

Consider the operator D_x^2 with (x,ξ) the relative coordinates. This operator clearly has symbol ξ^2. Now let us consider a change of coordinates to (y,η) given by $\phi(x) = y$ then in our new coordinates our operator becomes

$$(-id/dx)^2 = (-id/dx)(-id/dx) = (-id/dx)(-i\dot{\phi}d/dy) = (\ddot{\phi}d/dy - (\dot{\phi})^2(d/dy)^2$$

The symbol in new coordinates is then $(\dot{\phi})^2\eta^2 + i\dot{\phi}\eta$ which has lower order terms. Hence we can see that Ψ^∞ is not graded, but only filtered.

Proof. First we'll consider the $*$-algebra part. If

$$A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

then

$$A^* : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

is defined by duality.

$$A^*u(\bar{\phi}) = u(\overline{A\phi}) \;\; \forall \phi \in \mathcal{S}(\mathbb{R}^n) \tag{1.3.9}$$

We get the easy result if $u \in \mathcal{S}'(\mathbb{R}^n)$ then $A^*u \in \mathcal{S}'(\mathbb{R}^n)$ since $\psi \mapsto \overline{A\overline{\psi}} \in \mathcal{S}(\mathbb{R}^n)$ is continuous. Now let's look at this in terms of kernels.

$$\int K_{A^*}(x,y)u(y)\overline{\phi(x)}dydx = \int \overline{K_A(x,y)\phi(y)}dyu(x)dx \implies K_{A^*}(x,y) = \overline{K_A(y,x)} \tag{1.3.10}$$

What we immediately get is that if $K_A(x,y) = I(a(x,y,\xi))$ then $K_{A^*}(x,y) = I(\overline{a}(y,x,\xi))$ hence we are vindicated in our use of x and y. More importantly we see that interchanging x with y doesn't change the order of the symbol, nor does complex conjugation so in the end we get that

$$A^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \;\; when \;\; A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

Now onto the additivity of orders. We will apply reduction several times here. First we'll need it to write $A \in \Psi^m$ with $a(x,\xi)$. Second, we apply it to B^*. Notice what this will do. If the symbol for B^* is $\overline{b}(x,\xi)$ then the symbol for B by our above calculation is $b(y,\xi)$. Let's see this in action with a test function.

$$B^*u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\overline{b}(x,\xi)\hat{u}(\xi)d\xi$$

then

$$\langle B\phi, u\rangle = \langle \phi, B^*u\rangle = (2\pi)^{-n} \iint e^{-ix\cdot\xi}\phi(x)b(x,\xi)\overline{\hat{u}(\xi)}d\xi dx \implies \widehat{B\phi}(\xi) = \int e^{-iy\cdot\xi}b(y,\xi)\phi(y)dy \tag{1.3.11}$$

Now we see

$$AB(u) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi}a(x,\xi)b(y,\xi)u(y)dyd\xi$$

But we have that the product of symbols is additive in order. That is to say $a(x,\xi)b(y,\xi) \in S^{m+m'}(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi)$ hence $AB \in \Psi^{m+m'}(\mathbb{R}^n)$ which indeed is the result we want. □

1.4. **A Brief note on Quantization and Principal Symbols.** Since we have established reduction we now know that in the special case $w = 0$ and for now when $\partial_y a \equiv 0$ there are isomorphisms

$$\begin{aligned} \Psi^m(\mathbb{R}^n) &\xleftarrow{q_L} S^m(\mathbb{R}^n;\mathbb{R}^n) \\ \Psi^m(\mathbb{R}^n) &\xrightarrow{\sigma_L} S^m(\mathbb{R}^n;\mathbb{R}^n) \end{aligned} \tag{1.4.1}$$

Where $q_L = I$. This is called the *left quantization* of a. Its inverse σ_L is called the *full left symbol map.* Naively the correspondence looks something like

$$P(x,D) \leftrightarrow P(x,\xi).$$

Notice however, that in reduction we reduced $a(x,y,\xi) \rightsquigarrow a(x,\xi)$ and that this could define a ΨDO A, but when we looked at B^* in proving reduction its kernel was determined by $b(y,\xi)$. So there is no reason to think only about *left* quantization, we also have *right* quantization.

Definition 14. The *right quantization* of a symbol $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ is given by

$$q_R(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(y,\xi)d\xi \tag{1.4.2}$$

Notice again that $a(x,y,\xi) \rightsquigarrow a(y,\xi)$ instead of $a(x,\xi)$. Which leads us to the fact that there are also isomorphisms

$$\begin{aligned}\Psi^m(\mathbb{R}^n) &\xleftarrow{q_R} S^m(\mathbb{R}^n;\mathbb{R}^n)\\ \Psi^m(\mathbb{R}^n) &\xrightarrow{\sigma_R} S^m(\mathbb{R}^n;\mathbb{R}^n)\end{aligned} \tag{1.4.3}$$

There is no need to stop with only *left* and *right* quantization. We can define a whole continuous family given by

$$q_t(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a((1-tx+ty,\xi)d\xi \tag{1.4.4}$$

Which in exactly one small circle of people is referred to jokingly as the *GCA et al 2006 quantization.* From which we derive several very special cases.

1. $q_0 = q_L$
2. $q_1 = q_R$
3. $q_{1/2} = q_W$ the *Weyl quantization*

Remark 15. We have a somewhat cryptic correspondence between all these quantizations, given by

$$q_t = * \cdot q_{1-t} \cdot *$$

and in this case the right $*$ means adjoint on operators and the $*$ on the left means complex conjugation. This indeed shows us why *Weyl quantization* is special. It produces a self adjoint ΨDO.

Furthermore it should be noted that left quantization sends $x\xi \to xD$, right quantization $x\xi \to Dx$, and Weyl quantization $x\xi \to \frac{xD+Dx}{2}$. One final interesting remark is that in the Weyl algebra $\mathbb{C}[x,y]/([x,y]-1)$ we can represent y as $-d/dx$ or x as d/dy and so we have $d/dy(y\cdot)-y(d\cdot/dy) = 1$ and then the Weyl quantization of $x\xi$ gives $xD + 1/2$. This is guaranteed to be a self adjoint operator and in fact in quantum mechanics might then play the role of an observable. Indeed it is and partially accounts for the ground state energy in the quantum harmonic oscillator.

At this point we have another small annoyance to deal with. We have the fact that $\sigma_L(q_L(a))(x,\xi) = a(x,\xi)$, but $\sigma_R(q_L(a))(x,\xi) \neq a(x,\xi)$ We deal with this annoyance by the following lemma.

Lemma 16. *For any* $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ *we have*

$$\sigma_L(q_R(a))(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}D_x^\alpha D_\xi^\alpha a(x,\xi) \sim e^{i\langle D_x,D_\xi\rangle}a \tag{1.4.5}$$

Furthermore, by reversing the roles of L and R. we have

$$\sigma_R(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}D_\xi^\alpha D_x^\alpha \sigma_L(x,\xi) \sim e^{i\langle D_x,D_\xi\rangle}\sigma_L(x,\xi)$$

Proof. We can just apply the reduction theorem and keep track of our variables. □

We are in a position to define principal symbol. First we must realize that from the lemma we get

$$D_x^\alpha D_\xi^\alpha a(x,\xi) \in S^{m-|\alpha|}(\mathbb{R}^n;\mathbb{R}^n).$$

And then for any $A \in \Psi^m(\mathbb{R}^n)$ we have

$$\sigma_L(A) - \sigma_R(A) \in S^{m-1}(\mathbb{R}^n;\mathbb{R}^n).$$

Let us consider the algebraic quotient

$$S^{[m]}(\mathbb{R}^n;\mathbb{R}^n) = S^m(\mathbb{R}^n;\mathbb{R}^n)/S^{m-1}(\mathbb{R}^n;\mathbb{R}^n) \tag{1.4.6}$$

With $[a] \in S^{[m]}(\mathbb{R}^n;\mathbb{R}^n)$ denoting the image of $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$.

Definition 17. The *principal symbol* of $A \in \Psi^m(\mathbb{R}^n)$ is given by the principal symbol map

$$\sigma_m : \Psi^m(\mathbb{R}^n) \to S^{[m]}(\mathbb{R}^n;\mathbb{R}^n)$$

defined as

$$\sigma_m(A) = [\sigma_L(A)] = [\sigma_R(A)] \tag{1.4.7}$$

Lemma 18. *For every $m \in \mathbb{R}$ we have a short exact sequence*

$$0 \to \Psi^{m-1}(\mathbb{R}^n) \hookrightarrow \Psi^m(\mathbb{R}^n) \overset{\sigma_m}{\to} S^{[m]}(\mathbb{R}^n;\mathbb{R}^n) \to 0 \tag{1.4.8}$$

Proof. There is essentially nothing to show except that

$$\Psi^{m-1}(\mathbb{R}^n) \hookrightarrow \Psi^m(\mathbb{R}^n)$$

is injective and that

$$\Psi^m(\mathbb{R}^n) \overset{\sigma_m}{\to} S^{[m]}(\mathbb{R}^n;\mathbb{R}^n)$$

is surjective and has kernel $\Psi^{m-1}(\mathbb{R}^n)$. The first is taken care of by definition or order. The second is taken care of by the definition of principal symbol. □

We've shown that $\Psi^m \cdot \Psi^{m'} \subset \Psi^{m+m'}$. Let us list a few important properties of the principal symbol.

Lemma 19. *If $A \in \Psi^m(\mathbb{R}^n)$ and $B \in \Psi^{m'}(\mathbb{R}^n)$ then $A \circ B \in \Psi^{m+m'}(\mathbb{R}^n)$ and*

$$\sigma_{m+m'}(A \circ B) = \sigma_m(A) \cdot \sigma_{m'}(B) \tag{1.4.9}$$

$$\sigma_L(A \circ B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha \sigma_L(A) \cdot D_x^\alpha \sigma_L(B) \tag{1.4.10}$$

Before going on to the proof notice that we have a way of assimilating two ΨDO's even if $A = q_L(a)$ and $B = q_R(b)$ since we have an asymptotic sum telling us how to move from right quantization to left symbol.

Proof. The strategy of the proof is to write a lot of asymptotic sums and then regroup terms and apply Taylor's theorem to achieve our second formula.

The first equation is merely a consequence of

$$\sigma_L(A \cdot B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha [a(x,\xi) D_x^\alpha b(x,\xi)].$$

When $A = q_L(a)$ for $a \in S^m$, and $B = q_R(b)$ for $b \in S^{m'}$. We simply recall the definition of principal symbol and realize that we're only picking off the highest order term in the summation. That is to say we only consider $\alpha = 0$ for when $|\alpha| \geq 1$ it is clear that $m + m' - |\alpha| \leq m + m' - 1$.

Now we can turn to our asymptotic sums. Considering our situation of left and right quantizations and moving between left and right symbols we get

$$\sigma_L(A \circ B) \sim \sum_\beta \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha [\sigma_L(A) D_x^\alpha \frac{i^{|\beta|}}{\beta!} D_x^\beta D_\xi^\beta \sigma_L(B)].$$

Which is still aymptotically convergent. And now for the fun part! We need to regroup terms and relabel things in a clever way so as to appeal to Leibniz's formula and end up with our desired formula. Luckily for us Melrose has already done this. Let $\gamma = \alpha + \beta$ then

$$\sigma_L(A \circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} \sum_{0 \le \alpha \le \gamma} \frac{\gamma!(-1)^{|\gamma-\alpha|}}{\alpha!(\gamma-\alpha)!} D_\xi^\alpha [\sigma_L(A) \times D_x^{\gamma-\alpha} D_\xi^\gamma \sigma_L(B)].$$

Then remembering Leibniz's formula

$$d^n(fg) = \sum_{i+j=n} \begin{pmatrix} n \\ j \end{pmatrix} (d^i f) \cdot (d^j g).$$

We can regroup our double sum as

$$\sigma_L(A \circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} D_\xi^\gamma \sigma_L(A) \cdot D_x^\gamma \sigma_L(B)$$

$$\sim e^{i\langle D_y, D_\xi \rangle} \sigma_L(A)(x,\xi) \cdot \sigma_L(B)(y,\eta)|_{y=x,\xi=\eta}$$

□

Remark 20. Notice how much simpler the principal symbol is to deal with. We only need the top order term and when we compose ΨDO we get a multiplicative homomorphism instead of some horrible asymptotic sums. Of course what we lose is a lot of information. Luckily most of this information in expendable. In fact in the famous Atiyah-Singer index theorem the index of a differential operator is given in terms of it's principal symbol and the topology of the manifold on which it is acting. In essence, we can still say plenty about the existence to solutions of differential equations only considering the principal symbol.

1.5. **Ellipticity and Parametrices.**

One might wonder why I (or Melrose) went on a long diatribe about quantization and symbol maps and principal symbols. The answer will become apparent shortly. We are about to embark on a short but potent journey into elliptic ΨDO. If $A \in \Psi^m(\mathbb{R}^n)$ is elliptic then we like A very much. Ellipticity guarantees some nice properties for us, and also provides us with many tools for dealing with PDE. Of course I'm not a PDE person, but I can certainly understand why a systematic study of elliptic operators is useful.

As always we need to deal with symbols a little bit before we can get to pseudodifferential operators. Recall the following

Definition 21. A symbol $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ is said to be *elliptic* if

$$|a(x,\xi)| \ge \varepsilon(1+|\xi|)^m) \ \ in \ \ |\xi| \ge C_\varepsilon, \ \varepsilon > 0$$

Lemma 22. *If* $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ *is elliptic then* $\exists b \in S^{-m}(\mathbb{R}^n;\mathbb{R}^n)$ *such that*

$$a \cdot b - 1 \in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n) \tag{1.5.1}$$

The proof basically says $b = \frac{1}{a}$ outside a small neighborhood of the origin. And we have some special ways to cut it off near the origin.

Now we'll move on to elliptic ΨDO. We say $A \in \Psi^m(\mathbb{R}^n)$ is elliptic if it is almost invertible. Let us make this more formal.

Definition 23. Let $A \in \Psi^m(\mathbb{R}^n)$ then we say A is *elliptic* if and only if

$$\exists B \in \Psi^{-m}(\mathbb{R}^n) \ \ s.t. \ \ A \circ B - Id \in \Psi^{-\infty}(\mathbb{R}^n)$$

The meaning of almost invertible is now clear in this context. We mean invertible up to some additive smoothing operator. Or equivalently A is invertible in $\Psi^m(\mathbb{R}^n)/\Psi^{-\infty}(\mathbb{R}^n)$ with inverse in $\Psi^{-m}(\mathbb{R}^n)/\Psi^{-\infty}(\mathbb{R}^n)$. The big point is that ellipticity relies solely on the principal symbol.

The rest of this section is devoted to the big theorem and a few consequences of it.

Theorem 24. *Let $A \in \Psi^m(\mathbb{R}^n)$. Then the following are equivalent*

1. *A is elliptic*
2. *$\exists [b] \in S^{[-m]}(\mathbb{R}^n;\mathbb{R}^n)$ s.t. $\sigma_m(A) \cdot [b] \equiv 1 \in S^{[0]}(\mathbb{R}^n;\mathbb{R}^n)$*
3. *$\exists b \in S^{-m}(\mathbb{R}^n;\mathbb{R}^n)$ s.t. $\sigma_L(A) \cdot b - 1 \in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$*
4. *$\exists \varepsilon > 0$ s.t. $|\sigma_L(A)(x,\xi)| \geq \varepsilon(1+|\xi|)^m$ in $|\xi| > 1/\varepsilon$*

Proof. For (1) $\implies$ (2) we recall the homomorphism property of principal symbols. Let A and B be as in the definition of ellipticity then

$$\sigma_m(A) \cdot \sigma_{-m}(B) \equiv 1 \in S^{[0]}(\mathbb{R}^n;\mathbb{R}^n).$$

Then we take $[b] = \sigma_{-m}(B)$.

To show (2) $\Longrightarrow$ (3) recall that $\sigma_m(A) = [\sigma_L(A)]$. Pick a representative b_1 of $[b]$ then we have

$$\sigma_L(A) \cdot b_1 = 1 + e_{-1}, \ \ e_{-1} \in S^{-1}(\mathbb{R}^n;\mathbb{R}^n) \tag{1.5.2}$$

What we've essentially done is produced an element b_1 that insures $\sigma_L(A)$ is invertible in $S^{[m]}(\mathbb{R}^n;\mathbb{R}^n)$. Our task however is to knock the degree all the way down to $-\infty$ so that we get invertibility of A in $\Psi^m(\mathbb{R}^n)/\Psi^{-\infty}(\mathbb{R}^n)$. We can accomplish this by setting

$$f \sim \sum_{j\geq 1}(-1)^j e_{-1}^j \tag{1.5.3}$$

This is asymptotically convergent and the top order term of f is -1. Now let's check

$$(1+f)\cdot(1+e_{-1}) = 1 + e_{-1} - e_{-1} + \cdots = 1 + e_{-\infty}, \ \ e_{-\infty} \in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n) \tag{1.5.4}$$

So we have

$$\sigma_L(A) \cdot (b_1(1+f)) = 1 + e_{-\infty} \tag{1.5.5}$$

Setting $b = b_1(1+f)$ we achieve (3).

Since $e_{-\infty} \in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$ we have the estimate

$$\sup(1+|\xi|)^N |e_{-\infty}| < \infty \ \ \forall N$$

then certainly

$$\exists C \ \ s.t. \ \ |e_{-\infty}(x,\xi)| < \frac{1}{2} \ \ when \ \ |\xi| > C.$$

Thus

$$|\sigma_L(A)(x,\xi)| \cdot |b(x,\xi)| \geq \frac{1}{2} \ , \ |\xi| > C \tag{1.5.6}$$

and since $|b(x,\xi)| \leq C(1+|\xi|)^{-m}$ we have

$$\inf_{|\xi| \geq C} |\sigma_L(A)(x,\xi)| \cdot (1+|\xi|)^{-m} \geq C > 0. \tag{1.5.7}$$

Hence we have (4). By our lemma on elliptic symbols we have (4) $\implies$ (3).

Finally we wish to show (3) $\implies$ (1). Suppose (3) holds and let $B_1 = q_L(b)$. Then we have

$$\sigma_0(A \circ B_1) = \sigma_m(A) \cdot [b] \equiv 1$$

That is to say

$$A \circ B_1 - Id = E_{-1} \in \Psi^{-1}(\mathbb{R}^n).$$

Notice again that we're only guaranteed to have an operator of order -1 since that is what the principal symbol tells us. We will now perform the same trick as before to knock down the order to $-\infty$.

$$F \sim \sum_{j \geq 1} (-1)^j E_{-1}^j \tag{1.5.8}$$

Again since the top order of F is -1 we have $F \in \Psi^{-1}(\mathbb{R}^n)$. Then

$$(1+E_{-1})(1+F) = Id + E_{-\infty}, \ \ E_{-\infty} \in \Psi^{-\infty}(\mathbb{R}^n) \tag{1.5.9}$$

So we set $B = B_1(1+F)$ and we see that $A \circ B - Id = E_{-\infty} \in \Psi^{-\infty}(\mathbb{R}^n)$ as desired. $\square$

Remark 25. The entire proof we've just seen has been using the idea of a *right parametrix.* Of course this begs the question of whether or not we have a *left parametrix.* Of course we do, the next question that comes up is how these two are related.

Lemma 26. *Let $A \in \Psi^m(\mathbb{R}^n)$. Then A is elliptic $\iff$ there is a $B' \in \Psi^{-m}(\mathbb{R}^n)$ such that*

$$B' \circ A - Id = E' \in \Psi^{-\infty}(\mathbb{R}^n) \tag{1.5.10}$$

Furthermore if B is a right parametrix for A then $B - B' \in \Psi^{-\infty}(\mathbb{R}^n)$

Proof. The same construction as before holds when we realize $\sigma_{-m}(B') \cdot \sigma_m(A) \equiv 1$ hence

$$B' \circ A - Id = E'_{-1} \in \Psi^{-1}(\mathbb{R}^n)$$

And defining F' analogously

$$F' \sim \sum_{j \geq 1} (-1)^j (E'_{-1})^j$$

We set $B = (1+F') \circ B'$ and get that B' is indeed a left parametrix.

To show $B - B' \in \Psi^{-\infty}(\mathbb{R}^n)$ Consider

$$B' \circ (1+E_{-\infty}) = B' \circ (A \circ B) = (1+E'_{\infty}) \circ B$$

So we have $B' - B = E'_{-\infty}B - E_{-\infty}B' \in \Psi^{-\infty}(\mathbb{R}^n)$ $\square$

Remark 27. We've just seen that a parametrix is both a left and a right parametrix and is unique up to some additive term in $\Psi^{-\infty}$. In the land of symbols and quantization if $a \in S^m$ is elliptic and $A = q_t(a)$ then a parametrix for A is given by quantizing the symbol $b \in S^{-m}$ which is essentially $b = 1/a$. From here we see that in the pseudodifferential calculus is makes sense to speak of expressions like $\frac{1}{\langle D\rangle^m}$. Where $\langle D\rangle = (1 + D^2)^{1/2}$, and this makes sense too.

1.6. **Elliptic regularity.** The main point about elliptic regularity is that if $A \in \Psi^m(\mathbb{R}^n)$ is elliptic and $f \in H^k(\mathbb{R}^n)$ then

$$Au = f \implies u \in H^{k+m}(\mathbb{R}^n) \tag{1.6.1}$$

That is to say that solutions to elliptic equations pick up m orders of regularity (in Sobolev space) by virtue of a parametrix. Let's take a look at a special case.

Lemma 28. *If $A \in \Psi^m(\mathbb{R}^n)$ is elliptic and $Au = 0$ then $u \in C^\infty$.*

Proof. Since A is elliptic it has a (left) parametrix B such that $BA+E = Id$ where $E \in \Psi^{-\infty}(\mathbb{R}^n)$ Then

$$u = (BA + E)u = BAu + Eu = Eu \in C^\infty \tag{1.6.2}$$

Since E is a smoothing operator. □

The more general result is the following

Proposition 29. *If $A \in \Psi^m(\mathbb{R}^n)$ is elliptic then*

$$\begin{aligned} Au \in \langle x\rangle^p H^q(\mathbb{R}^n),\ u \in \langle x\rangle^{p'} H^{q'}(\mathbb{R}^n) \\ \implies u \in \langle x\rangle^{p"} H^{q"}(\mathbb{R}^n), \\ p" = \max(p, p'),\ q" = \max(q + m, q') \end{aligned} \tag{1.6.3}$$

I'll give the proof and then go back to fill in a few details about Sobolev spaces.

Proof. Let B be a left parametrix for A. Then

$$BA = Id + G,\ G \in \Psi^{-\infty}(\mathbb{R}^n)$$

Then

$$u = B(Au) + Gu \in \langle x\rangle^p H^{q+m}(\mathbb{R}^n) + \langle x\rangle^{p'} H^\infty(\mathbb{R}^n) \subset \langle x\rangle^{p"} H^{q+m}(\mathbb{R}^n) \tag{1.6.4}$$

And we see in the special case $p, p' = 0$ then we pick up m orders of regularity without the Tom Foolery. □

The Sobolev spaces $H^k(\mathbb{R}^n)$ are defined as subspaces of $L^2(\mathbb{R}^n)$ for positive k. More specifically

Definition 30. For any positive integer k we have

$$H^k(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}^n)\ \forall|\alpha| \le k\}$$

Of course we have some nice properties involving Fourier transforms and see

$$u \in H^k(\mathbb{R}^n) \implies \xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n)\ \ \forall|\alpha| \le k.$$

Combining all these fun conditions we get

$$u \in H^k(\mathbb{R}^n) \implies (1 + |\xi|^2)^{k/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\ \ \forall|\alpha| \le k.$$

Notice now that $a(\xi) = (1 + |\xi|^2)^{k/2} \in S^k(\mathbb{R}^n; \mathbb{R}^n)$. Again by $\langle\xi\rangle$ we mean $(1 + |\xi|^2)^{1/2}$.

Quantizing our $\langle\xi\rangle^k$ we give meaning to $\langle D\rangle^k$. Our new condition is that

$$u \in H^k(\mathbb{R}^n) \iff \langle D\rangle^k u \in L^2(\mathbb{R}^n) \tag{1.6.5}$$

Moreover this is the correct definition for $k \in \mathbb{R}$!

A point of interest for those interested in PDE.

$$\|u\|_m^2 = \|\langle D\rangle^m u\|_{L^2}^2 = \int (1+|\xi|^2)^m |\hat{u}(\xi)|^2 d\xi \tag{1.6.6}$$

Let's reel off a few results and that should be sufficient for understanding elliptic regularity as above.

$$H^m(\mathbb{R}^n) \subset H^{m'}(\mathbb{R}^n)\ if\ m' \leq m.$$

We also have our two extreme spaces

1. $H^\infty(\mathbb{R}^n) = \bigcap_m H^m(\mathbb{R}^n)$ (the residual space)
2. $H^{-\infty}(\mathbb{R}^n) = \bigcup_m H^m(\mathbb{R}^n)$ (the big space)

Now we have a string of inclusions

$$\mathcal{S}(\mathbb{R}^n) \subsetneq H^\infty(\mathbb{R}^n) \subsetneq H^{-\infty}(\mathbb{R}^n) \subsetneq \mathcal{S}'(\mathbb{R}^n) \tag{1.6.7}$$

We can see the inclusions are proper by noting

1. $\frac{1}{1+x^2} \in H^\infty(\mathbb{R})$
   $\frac{1}{1+x^2} \notin \mathcal{S}(\mathbb{R})$
2. $1 \in H^{-\infty}(\mathbb{R})$
   $1 \notin H^\infty(\mathbb{R})$
3. $x \in \mathcal{S}'(\mathbb{R})$
   $x \notin H^{-\infty}(\mathbb{R})$

Since we don't capture all the tempered distributions we consider *weighted Sobolev spaces.*

$$\langle x\rangle^q H^m(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle x\rangle^{-q} u \in H^m(\mathbb{R}^n)\} \tag{1.6.8}$$

Theorem 31. *For each $q, m, M \in \mathbb{R}$ each $A \in \Psi^M(\mathbb{R}^n)$ defines a continuous linear map*

$$A : \langle x\rangle^q H^m(\mathbb{R}^n) \longrightarrow \langle x\rangle^q H^{m-M}(\mathbb{R}^n)$$

Lemma 32.

$$\mathcal{S}'(\mathbb{R}^n) = \bigcup_M \langle x\rangle^M H^{-M}(\mathbb{R}^n)$$

2. Hochschild and Cyclic Homology Basics

2.1. **Definitions.**

The standard definitions of Hochschild and Cyclic Homology involve references to a ground ring k. I however will use $\mathbb{C}$ unless otherwise noted.

Definition 33. Let A be a $\mathbb{C}$-algebra and let M be an A bimodule. Then we define the space $C_n(A, M) := M \otimes A^{\otimes n}$ where our tensor is over $\mathbb{C}$. These are the *Hochschild chains* of degree n.

As with any homology theory we have $C_*(A, M)$:

$$\cdots C_{n+1}(A, M) \xrightarrow{b} C_n(A, M) \xrightarrow{b} C_{n-1}(A, M) \cdots$$

Now it is our job to give a map, which is called b, between these spaces and make sure that b^2=0. Then we can proceed with the homology.

Definition 34. The map $b' : C_{n+1}(A, M) \to C_n(A, M)$ is given by

$$b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots a_n)$$

It is important to note that in the above definition $a_0 \in M$. Our desired map b is given by

$$b(a_0 \otimes \cdots \otimes a_n) = b'(a_0 \otimes \cdots \otimes a_n) + (-1)^n (a_n a_0 \otimes \cdots \otimes a_{n-1})$$

.

The reason we give the map b' first is because when our algebra A is unital then $(C_*(A, M), b')$ is acyclic. We can see this explicitly by constructing a chain homotopy from *id* to 0.

Remark 35. if two maps are chain homotopic then they induce the same map on homology. We can see this by looking directly at cycles.

Definition 36. Let $C_\bullet$ and $D_\bullet$ be two chain complexes. Two maps $f, g : D_\bullet \to C_\bullet$ are *chain homotopic* if there exists a map $T : D_{n-1} \to C_n$ such that
$\partial T + T\partial = f - g$.

This can be seen by the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] & C_{n-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] \arrow[ul, "T"] & D_{n-1} \arrow[r] \arrow[ul, "T"] & \cdots
\end{tikzcd}$$

Let us see why $f_* = g_*$ on homology.
Let z be a cycle then $\partial z = 0$ and

$$f_*(z) = g_*(z) + \partial T(z)$$

But $\partial T(z)$ is a boundary, hence $f_* = g_*$ on homology.

In the special case $A = M$ we can define a chain homotopy to show $(C_*(A, A), b')$ is acyclic.

Definition 37. Let $s : C_n(A, A) \to C_{n+1}(A, A)$ be defined by

$$s(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n$$

.

Lemma 38.

$$sb' + b's = id - 0$$

Hence when A is unital $id \simeq 0$

Proof. Direct computation. □

From here on out I will be in the special situation $A = M$ unless specifically noted otherwise. The Notation for $C_n(A, A)$ will now be $C_n(A)$.

Definition 39. The *bar complex* of A is $(C_*(A), b)$. The *acyclic bar complex* of A is $(C_*(A), b')$.

Notice that $C_n(A) := A^{\otimes n+1}$.

2.2. **Examples and Theorems.** It is important to realize that Hochschild Homology is a homology theory and so we have lots of goodies to go along with that. We'll denote $H_*(C_\bullet(A), b)$ as $HH_*(A)$. This happens to be functorial in A.

Without further ado I'll get to some examples.

Example 40. The first and easiest example is $HH_0(A)$.

$$b : A \otimes A \to A$$

is given by

$$b(a_0 \otimes a_1) = a_0a_1 - a_1a_0$$

So we see that

$$HH_0(A) = ker(b)/Im(b) = A/[A, A]$$

Example 41. Another important example is one where the "acyclic" bar complex has nontrivial homology. Here we are required to consider a nonunital algebra. Consider $A = \mathbb{C}[x]$ and $I = xA$ the ideal generated by x. Let us look at the map $b' : A \otimes A \to A$ which in this case is just the standard multiplication map.

$$b'(a_0 \otimes a_1) = a_0a_1$$

Then the kernel of b' is generated by $ax \otimes x - x \otimes ax$. In the unital case the kernel is generated by $a \otimes 1 - 1 \otimes a$, but we have no unit. Hence here the kernel contains I^2. Thus we have

$$HH_0(I) = I/I^2 = x\mathbb{C}[x]/(x^2) \cong \mathbb{C} \neq 0$$

Then $(C_*(I), b')$ is not acyclic.

Theorem 42. *$HH_n(A)$ is a module over $Z(A)$ the center of A.*

Proof. We can see this easily by looking at the chains. In fact $H_n(A, M)$ is a module over $Z(A)$ for any M. Let $z \in Z(A)$ then z acts on $C_n(A.m)$ by

$$z \cdot (m \otimes a_1 \otimes \cdots \otimes a_n) = (zm \otimes a_0 \otimes \cdots \otimes a_n)$$

It is clear that $z \cdot b(m \otimes a_1 \otimes \cdots \otimes a_n) = b(zm \otimes a_1 \otimes \cdots \otimes a_n)$ □

In particular we see that is A is commutative then HH_n is a module over A.

Example 43. This may be perhaps the most important example of this section. Let A be a smooth algebra then $HH_n(A) \cong \Omega^n_A$ by the Hochschild-Kostant-Rosenberg (HKR) map.

Proposition 44. *Let $\mathcal{A}$ be a unital and commutative algebra. Then there is a canonical isomorphism $HH_1(\mathcal{A}) \cong \Omega^1_{\mathcal{A}}$.*

Proof. Let us first examine the map $b : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$. We see by definition $b(a_0 \otimes a_1) = a_0a_1 - a_1a_0$ but since $\mathcal{A}$ is commutative this is the zero map. Therefore $HH_1(\mathcal{A})$ is simple the quotient of $\mathcal{A} \otimes \mathcal{A}$ by the relation

$$ab \otimes c - a \otimes bc + ca \otimes b = 0$$

If one were to stare at this relation for a small while (a year or so, maybe less is one is quick) one might realize that this relation can be satisfied by

$$a \otimes b \mapsto adb$$

where adb is the *Kähler differential*. On a smooth manifold this is the exterior derivative and the Kähler differential is simply the formal algebraic structure. That is $d(fg) = fdg + (df)g$. One can immediately check that this map is well defined and satisfies the relation for HH_1. □

Theorem 45. *More is true. When A is a smooth algebra (polynomial, symmetric, $C^\infty(M)$) then the previous result extends to all n. And indeed*

$$HH_*(A) \cong \Omega^*_A$$

is an isomorphism of graded algebras.

Proof. The only problem we might face in generalizing this map to higher homologies is torsion. We have avoided that by using smooth algebras. The HKR map in this case is

$$\chi(a_0 \otimes \cdots \otimes a_n) = \frac{1}{n!} a_0 da_1 \wedge \cdots \wedge da_n$$

□

Remark 46. BBN and Loday both provide somewhat more extensive proofs of this theorem.

Remark 47. Since we are dealing with homology, another useful point of view is that of the Tor functor. In the case of Hochschild Homology we have

$$H_n(A, M) = Tor_n^{A\otimes A^{op}}(A, M).$$

Example 48. Let $\mathbb{C}[\varepsilon] := \mathbb{C}[x]/(x^2)$ be the ring of dual numbers. Then we have the result $HH_n(\mathbb{C}[\varepsilon]) \cong \mathbb{C}[\varepsilon]$ for $n \geq 0$. The generators are given by $1 \otimes \varepsilon^{\otimes(2n+1)}$ for HH_{2n+1} and $\varepsilon \otimes \varepsilon^{2n}$ for HH_{2n}. We can see this by taking the free resolution of $\mathbb{C}[\varepsilon]$ to be

$$\cdots \to \mathbb{C}[\varepsilon] \xrightarrow{\varepsilon} \mathbb{C}[\varepsilon] \xrightarrow{\varepsilon} \mathbb{C}[\varepsilon] \to \cdots$$

Then is it a routine matter of homological algebra to compute $Tor_n^{\mathbb{C}[\varepsilon]\otimes\mathbb{C}[\varepsilon]^{op}}(\mathbb{C}[\varepsilon], \mathbb{C}[\varepsilon])$

Although not yet exploited in these notes the idea of computing homology groups as Tor groups is very powerful. Let us recall from basic homological algebra that any two homotopic chain complexes produce the same homology groups. Furthermore Tor is symmetric in its variables therefore we can take a resolution of either algebra. In my case I'll usually be considering both variables to be the same algebra. Finally it should be mentioned that the Koszul type resolutions appear often in the calculation of Hochschild homology groups when using the *Tor* approach.

Lemma 49. *Let $ad(u) : C_n(A) \to C_n(A)$ be given by*

$$ad(u)(a_0 \otimes \cdots \otimes a_n) = \sum_{0 \le i \le n} (a_0 \otimes \cdots \otimes [u, a_i] \otimes \cdots \otimes a_n)$$

Then $ad(u)$ is chain homotopic to zero.

Proof. Let $h(u) : C_n(A) \to C_{n+1}(A)$ be given by

$$h(u)(a_0 \otimes \cdots \otimes a_n) := \sum_{0 \le i \le n} (-1)^i (a_0 \otimes \cdots \otimes a_i \otimes u \otimes \cdots \otimes a_n)$$

then $h(u)$ provides the chain homotopy i.e.

$$bh(u) + h(u)b = -ad(u)$$

Which is to say $ad(u)_* : HH_n(A) \to HH_n(A)$ is the zero map. □

Example 50. Let us use $ad(u)$ to show $HH_0(W) = 0$ where $W = \mathbb{C}[x, y]/([x, y] - 1)$ is the *Weyl algebra* in one dimension.

We know the $ad(u)_* \simeq 0$ for any u. So let's just choose a clever $u \in W$ and show that $ad(u)_* \simeq id$ on $HH_0(W)$.

By earlier computations we know $HH_0(W) = W/[W, W]$ so

$$ad(u)_* : W/[W, W] \to W/[W, W]$$

is given by

$$ad(u)(v + [W, W]) = [u, v] + [W, W].$$

Let us choose $u = x$ and apply $ad(x)_*$ to $\overline{y} = y + [W, W] \in W/[W, W]$ to see

$$ad(x)(y + [W, W]) = [x, y] + [W, W] = 1 + [W, W] = 0 + [W, W]$$

Hence $1 \in [W, W]$ so $ad(x)_*$ induces the identity map on $HH_0(W)$ but we know $ad(x)_* \simeq 0$ so $HH_0(W) = 0$

Remark 51. A more important fact is that $ad(u)$ is an inner derivation, and any inner derivation can be written $ad(u)$ for some u. The point being that any map induced on $HH_n(A)$ by an inner derivation is always a zero map.

2.3. Definitions of Cyclic Homology.

Cyclic homology has several different origins. Connes and Tsygan developed this concept independently in the early 80's. Connes' theory is denoted H_n^λ and is gotten by using the standard Hochschild differential on the complex

$$\cdots \to C_{n+1}^\lambda \to C_n^\lambda \to C_{n-1}^\lambda \to \cdots$$

where C_n^λ is the space of $\mathbb{Z}/(n+1)\mathbb{Z}$ coinvariant chains $(C_n)_{\mathbb{Z}/(n+1)\mathbb{Z}}$.

Tsygan's construction on the other hand views Cyclic homology as the homology of a bicomplex which I will attempt to define in this section.

Definition 52. The *cyclic operator* which we call t acts on elements of $A^{\otimes n+1}$ by

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes \cdots \otimes a_{n-1})$$

and then is extended to all of $A^{\otimes n+1}$ by linearity. The *norm operator* which we call $N = 1 + t + \cdots + t^n$ also acts on $A^{\otimes n+1}$

Notice that $(1-t)N = 1 - t^{n+1} = 0$.

Lemma 53. *The operators t, N, b', b satisfy the following*

$$(1-t)b' = b(1-t),\ Nb = b'N$$

Definition 54. The *Cyclic Bicomplex of A* denoted $CC(A)$ is the following thing

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & {} \arrow[l, "N"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & {} \arrow[l, "N"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & A \arrow[l, "1-t"'] & {} \arrow[l, "N"']
\end{tikzcd}$$

By convention the module in the bottom left hand corner is of bidegree $(0,0)$ so that $CC_{p,q}(A) = C_q(A) = A^{\otimes q+1}$.

Definition 55. The *cyclic homology groups* $HC_n(A), n \geq 0$ are the homology groups of the total complex $Tot(CC(A))$. That is

$$HC_n(A) := H_n(Tot(CC(A)))$$

Remark 56. Let us quickly note the following

1. $Tot(CC(A))_n = \bigoplus_{p+q=n} CC_{p,q}(A)$
2. The even columns of $CC(A)$ are Hochschild complexes
3. The odd columns are acyclic bar resolutions

As briefly mentioned previously Connes has a complex for cyclic homology denoted $C_*^\lambda(A)$ with corresponding homology $H_n^\lambda(A)$. When the ground ring $k \supseteq \mathbb{Q}$ then $H_n^\lambda(A) \cong HC_n(A)$, but of course in my case $k = \mathbb{C}$ so this is indeed an isomorphism. More formally;

Definition 57. *Connes' complex* $C_*^\lambda(A)$ is given by

$$C_*(A):\ \cdots \xrightarrow{b} C_{n+1}^\lambda(A) \xrightarrow{b} C_n^\lambda(A) \xrightarrow{b} C_{n-1}^\lambda(A) \xrightarrow{b} \cdots \xrightarrow{b} C_0^\lambda(A)$$

where $C_n^\lambda(A) := A^{\otimes n+1}/(1-t)$

The map p from $Tot(CC(A))$ to $C_*^\lambda(A)$ that produces the isomorphism on homology is given by

$$p(CC_{r,s}(A)) = \begin{cases} C_s^\lambda(A) & r = 0 \\ 0 & \text{else} \end{cases}$$

That is to say we map $A^{\otimes n+1} \mapsto A^{\otimes n+1}/(1-t)$ on the left most column and zero everywhere else.

In Loday's presentation of this topic he jumps from this to a lemma about killing contractible complexes. I feel this lemma is mostly tedious, but the result is somewhat intuitive and will lead to an important construction and therefore earns mention. It says essentially this

Lemma 58. *Let (A_*, δ) and (A'_*, δ') be two complexes such that (A'_*, δ') is acyclic. Then $H_n(A_* \oplus A'_*, \delta \oplus \delta') \cong H_n(A_*, \delta)$.*

We are now ready to define the map B and give the $(b, B)-$*bicomplex* definition of cyclic homology. By the lemma we see that the odd columns in $CC(A)$ are acyclic and thus the total homology is not affected by them. This is the source of the map B. Notice $CC_{2q}(A) = A^{\otimes q+1}$. We define B as follows.

$$B = (1-t)sN \tag{2.3.1}$$

So $B : CC_{p+2,q} \to CC_{p,q}$.

Our new monstrous complex looks like

$$\begin{tikzcd}
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[d, "b"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[d, "b"'] \arrow[ul, "B"] & A^{\otimes 2} \arrow[d, "b"'] \arrow[ul, "B"] \\
A & A \arrow[ul, "B"] & A \arrow[ul, "B"]
\end{tikzcd}$$

Which is traditionally rearranged to look like this

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"'] & A \arrow[l, "B"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd}$$

The (b, B) bicomplex is called $\mathcal{B}(A)$ and notice that

$$\mathcal{B}(A)_{p,q} = \begin{cases} A^{\otimes q-p+1} & q \geq p \\ 0 & \text{else} \end{cases}$$

Remembering our relationships

$$\begin{aligned} Nb &= b'N \\ (1-t)b' &= b(1-t) \\ b's + sb' &= id \end{aligned}$$

We can see that

$$bB + Bb = 0.$$

Let us see what B does explicitly.

$$\begin{aligned} B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} &(-1)^{ni}(1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}) \\ &-(-1)^{ni}(a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}) \end{aligned} \tag{2.3.2}$$

For $n = 0, 1$ we get

$$B(a_0) = 1 \otimes a_0 + a_0 \otimes 1$$

$$B(a_0 \otimes a_1) = (1 \otimes a_0 \otimes a_1 - 1 \otimes a_1 \otimes a_0) + (a_0 \otimes 1 \otimes a_1 - a_1 \otimes 1 \otimes a_0)$$

Since B is defined at the level of chains it induces a homomorphism on $HH_*(A)$ denoted by

$$B_* : HH_n(A) \to HH_{n+1}(A)$$

which is injective as a map of complexes $Tot(\mathcal{B}(A)) \hookrightarrow Tot(CC(A))$ defined by

$$\begin{aligned} C_{q-p} = \mathcal{B}(A)_{p,q} \ni x \mapsto x \oplus sN(x) &\in C_{q-p} \oplus C_{q-p+1} \\ &= CC_{2p,q-p} \oplus CC_{2p-1,q-p+1} \subset Tot(CC(A))_{p+q} \end{aligned} \tag{2.3.3}$$

Definition 59. If A is unital then the inclusion $Tot(\mathcal{B}(A)) \hookrightarrow Tot(CC(A))$ is a quasi-isomorphism. We then define $H_n(Tot(\mathcal{B}(A)) = HC_n(A)$. This is the (b, B) definition of cyclic homology.

As always in mathematics we try to make things simpler for ourselves rather than more complicated. So we can define the reduced complex $\overline{\mathcal{B}}(A)$ by replacing the Hochschild complexes with their renormalizations.

Definition 60. The bicomplex $\overline{\mathcal{B}}(A)$ is the following thing.

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
A \otimes \bar{A}^{\otimes 2} \arrow[d, "b"'] & A \otimes \bar{A} \arrow[l, "\bar{B}"'] \arrow[d, "b"'] & A \arrow[l, "\bar{B}"'] \\
A \otimes \bar{A} \arrow[d, "b"'] & A \arrow[l, "\bar{B}"'] & \\
A & &
\end{tikzcd}$$

Where $\bar{B} = sN : A \otimes \bar{A}^{\otimes n} \to A \otimes \bar{A}^{\otimes n+1}$ and is given by

$$\bar{B}(a_0 \otimes \cdots \otimes a_n) = \sum^{n} (-1)^{ni} (1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_1 \otimes \cdots \otimes a_{i-1}) \tag{2.3.4}$$

Notice the simplicity afforded in lower dimensions

$$\bar{B}(a_0) = 1 \otimes a_0,\ \bar{B}(a \otimes a') = 1 \otimes a \otimes a' - 1 \otimes a' \otimes a$$

Lemma 61. *The surjection $\mathcal{B}(A) \to \overline{\mathcal{B}}(A)$ is a quasi-isomorphism.*

Corollary 62. *We have "constructed" the following sequence*

$$Tot(\overline{\mathcal{B}}(A)) \leftarrow Tot(\mathcal{B}(A)) \hookrightarrow Tot(CC(A)) \to C^{\lambda}(A)$$

Where each map is a quasi-isomorphism so long as $k \supset \mathbb{Q}$, but again we are assuming that for now.

Now we'll move onto a somewhat surprising result; a long exact sequence relating HH_* with HC_*.

Theorem 63. *For any associative, not necessarily unital algebra A there is a natural long exact sequence.*

$$\cdots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} \cdots \tag{2.3.5}$$

Proof. Let $CC(A)^{\{2\}}$ be the bicomplex consisting of the first two columns of $CC(A)$ that is

$$CC(A)^{\{2\}}_{p,q} = \begin{cases} CC(A)_{p,q} & p = 0, 1 \\ 0 & \text{else} \end{cases}$$

Then it is clear that the following sequence is exact

$$0 \longrightarrow CC(A)^{\{2\}} \longrightarrow CC(A) \longrightarrow CC(A)[2,0] \to 0$$

Where $CC(A)[a,b]_{p,q} = CC(A)_{p-a,q-b}$.

The long exact exact in the theorem is exactly the long exact sequence in homology associated with the above exact sequence. We can see that $CC(A)^{\{2\}}$ is quasi-isomorphic to $C_*(A)$ (the Hochschild complex) as the second column is acyclic when A is unital, but more generally $CC(A)^{\{2\}}$ is the correct complex to use for $HH_*(A)$ even when A is not unital. Also we can easily see that the homology of $CC(A)[2,0]$ is the same as $CC(A)$. □

We call the map S the *periodicity operator*. In cyclic cohomology there is an analogous map S which has the effect of promoting any $n-$*cocycle* to an $n+2-$*cocyle* by means of a cup product.

Remark 64. When A is unital we have a slightly easier job in defining S. We can use the bicomplex $\mathcal{B}(A)$ and the sequence of complexes

$$0 \longrightarrow C(A) \longrightarrow Tot(\mathcal{B}(A)) \xrightarrow{S} Tot(\mathcal{B}(A))[2] \longrightarrow 0$$

Where S in this case is given by factoring out the first column of $\mathcal{B}(A)$. By recalling the construction of $\mathcal{B}(A)$ it is easy to see that these two maps (S for $CC(A)$ and $\mathcal{B}(A)$) agree.

Example 65. When $A = \mathbb{C}$ the map S sends the canonical generator $u^n \in HC_{2n}(\mathbb{C})$ to $u^{n-2} \in HC_{2n-2}(\mathbb{C})$ and in this case is an isomorphism.

Corollary 66. *If $f : A \to A'$ induces an isomorphism on Hochschild homology then it induces an isomorphism on cyclic homology.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & HH_n(A) \arrow[r, "I"] \arrow[d] & HC_n(A) \arrow[r, "S"] \arrow[d] & HC_{n-2}(A) \arrow[r, "B"] \arrow[d] & HH_{n-1}(A) \arrow[r, "I"] \arrow[d] & \cdots \\
\cdots \arrow[r] & HH_n(A') \arrow[r, "I"] & HC_n(A') \arrow[r, "S"] & HC_{n-2}(A') \arrow[r, "B"] & HH_{n-1}(A') \arrow[r, "I"] & \cdots
\end{tikzcd}$$

We already know that $HH_0(A) \cong HC_0(A)$ so applying the Five Lemma at low dimensions we can deduce that $HC_1(A) \cong HC_1(A')$ then by induction and repeated application of the Five Lemma we obtain $HC_n(A) \cong HC_n(A')$ which is our desired result. □

Let's end this section with one final important result useful for computing cyclic homology. This is the Kunneth theorem for cyclic homology.

Theorem 67. *If C'_* and $HC_*(C')$ are projective (for example over a field) then there is a long exact sequence.*

$$\cdots \to HC_n(C\otimes C') \xrightarrow{i} \bigoplus_{p+q=n} HC_p(C)\otimes HC_q(C') \xrightarrow{S\otimes 1 - 1\otimes S}$$

$$\bigoplus_{p+q=n-2} HC_p(C)\otimes HC_q(C') \xrightarrow{\partial} HC_{n-1}(C\otimes C') \to \cdots \tag{2.3.6}$$

2.4. **More Examples.** This section is dedicated to more examples of Hochschild homology and then examples of moving to cyclic homology from Hochschild. Let's first revisit one of our favorite examples.

Example 68. Here we'll revisit smooth algebras.

Theorem 69. *If A is smooth over $\mathbb{C}$ then there is a spectral sequence converging to cyclic homology whose E^2 term is given by*

$$E^2_{p,q} = \left\{ \begin{array}{ll} \Omega^q_A/d\Omega^{k-1}_A, & p=0 \\ H^{q-p}_{DR}(A), & p>0 \end{array} \right\} \implies HC_{p+q}(A)$$

A good question is "what breaks down when A is not smooth?" Perhaps a better question is "what happens when A is not commutative?" In principal we noncommutative algebras can have Hochschild and cyclic homologies isomorphic to these. A good example is the noncommutative torus which we'll visit later. The reason that we require smoothness is because it insures certain isomorphisms when we're working with the Tor definition of homology. We have a lemma from Loday that I won't prove, but bears mention.

Lemma 70. *Let R be commutative and I an ideal generated by a regular sequence. The the morphism $\varepsilon_* : \Lambda^*_{R/I} \to Tor^R_*(R/I, R/I)$ deduced from $\varepsilon_1 : I/I^2 \overset{\simeq}{\to} Tor^R_1(R/I, R/I)$ is an isomorphism of algebras.*

Using this lemma we can then construct a Koszul resolution which is a free resolution of R/I and then it is "easy" to compute the homology via Tor. However without smoothness we run into trouble with this resolution.

Back to the spectral sequence.

Proposition 71. *For any unital and commutative $\mathbb{C}$-algebra the following diagram is commutative*

$$\begin{tikzcd}
HH_n(A) \arrow[r, "B_*"] \arrow[d, "\pi_n"'] & HH_{n+1}(A) \arrow[d, "\pi_{n+1}"'] \\
\Omega^n_A \arrow[r, "(n+1)d"] & \Omega^{n+1}_A
\end{tikzcd}$$

Where $\pi_n(a_0\otimes\cdots\otimes a_n) = a_0 da_1 \ldots da_n$

Proof. We have a formula

$$\pi_{n+1}B(a_0\otimes\cdots\otimes a_n) = \sum_{i=0}^{n} (-1)^{ni} da_i \ldots da_n da_0 \ldots da_{i-1} = \sum_{i=0}^{n} da_0 \ldots da_n =$$

$$(n+1)da_0\ldots da_n = (n+1)d\pi_n(a_0\otimes\cdots\otimes a_n)$$

Hence we get the commutativity of

$$\begin{tikzcd}
C_n(A) \arrow[r, "B"] \arrow[d, "\pi_n"'] & C_{n+1}(A) \arrow[d, "\pi_{n+1}"'] \\
\Omega^n_A \arrow[r, "(n+1)d"] & \Omega^{n+1}_A
\end{tikzcd}$$

Then since π is a morphism of complexes our above complex is retrieved by taking homology. $\square$

Now we're in business since $(1/n!)\pi_n$ induces a map of complexes from $\mathcal{B}(A)$ to $\mathcal{D}(A)$ (The reduced Deligne complex or sometimes reduced de Rham complex) given by

$$\mathcal{D}(A): \begin{array}{ccccc} 0\downarrow & & 0\downarrow & & 0\downarrow \\ \Omega^n_A & \xleftarrow{d} & \Omega^1_A & \xleftarrow{d} & \Omega^0_A \\ 0\downarrow & & 0\downarrow & & \\ \Omega^1_A & \xleftarrow{d} & \Omega^0_A & & \\ 0\downarrow & & & & \\ \Omega^0_A & & & & \end{array}$$

This follows from the fact that $\pi_n \circ b = 0$. Now what we've done is to mix up the complexes $\mathcal{B}(A)$ and $\mathcal{D}(A)$ when A is smooth. This makes our computation easy when we compute the vertical differentials first. Then we see when computing horizontal differentials that we get exactly

$$E^2_{p,q} = \left\{ \begin{array}{ll} \Omega^q_A/d\Omega^{k-1}_A, & p=0 \\ H^{q-p}_{DR}(A), & p>0 \end{array} \right\}$$

Furthermore we've killed all our differentials so that

$$HC_n(A) = \bigoplus_{p+q=n} E^2_{p,q} = \Omega^n_A/d\Omega^{n-1}_A \oplus \bigoplus_k H^{n-2k}_{DR}(A) \tag{2.4.1}$$

Example 72. Let's quickly see how this follows from Connes' periodic exact sequence.

$$\begin{array}{ccccccccc}
\cdots \to & HH_n(A) & \xrightarrow{I} & HC_n(A) & \xrightarrow{S} & HC_{n-2}(A) & \xrightarrow{B} & HH_{n-1}(A) & \to \cdots \\
 & \| & & \| & & \| & & \| & \\
\cdots \to & \Omega^n_A & \to & \Omega^n_A/d\Omega^{n-1}_A & \to & 0 & & & \\
 & & & \oplus & & & & & \\
 & 0 & \to & H^{n-2}_{DR} & \to & \Omega^{n-2}_A/d\Omega^{n-3}_A & \to & \Omega^{n-1}_A & \to \cdots \\
 & & & \oplus & & \oplus & & & \\
 & & & H^{n-4}_{DR} & = & H^{n-4}_{DR} & & & \\
 & & & \oplus & & \oplus & & & \\
 & & & \cdots & = & \cdots & & &
\end{array}$$

In particular looking at the column under $HC_n(A)$ we see what we want to.

Let's see what this means for several particular algebras.

Example 73. Let $A = \mathbb{C}[x]$ then we have the following results

$$HH_n(A) = \begin{cases} \mathbb{C}[x] & n = 0 \\ \mathbb{C}[x] \cdot dx & n = 1 \\ 0 & \text{else} \end{cases}$$

$$HC_n(A) = \begin{cases} \mathbb{C}[x] & n \text{ even} \\ \mathbb{C}[x] & n \text{ odd} \end{cases}$$

Example 74. Let $A = \mathbb{C}[x_1, \ldots, x_k]$ Then we have the following.

$$HH_n(A) = \begin{cases} \mathbb{C}[x_1, \ldots, x_k] & n = 0 \\ \\ \bigoplus_1^k \mathbb{C}[x_1, \ldots, x_k] \cdot dx_i & n = 1 \\ \\ \bigoplus_1^{C(k,j)} \mathbb{C}[x_1, \ldots, x_k] \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_j} & n = j, i_1 < \cdots < i_j \\ \\ \mathbb{C}[x_1, \ldots, x_k] \cdot dx_1 \wedge \cdots \wedge dx_k & n = k \\ \\ 0 & n > k \end{cases}$$

We also can see that

$$H^n_{DR}(A) = \{\sum a(x_1, \ldots, x_k) dx_{i_1} \wedge \cdots \wedge dx_{i_n} |\, a \in A\}$$

Which is the same as $HH_n(A)$ so we get

$$HC_n(A) = \begin{cases} A & n = 0 \\ \\ \bigoplus_1^{many} A & n\text{-even} \\ \\ \bigoplus_1^{many} A & n\text{-odd} \end{cases}$$

Example 75. Let $A = C^\infty(S^1)$. This is a smooth algebra so our theorem applies and

$$HH_n(A) = \begin{cases} A & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

We need to note that in our notations $\Omega^n_M = \Omega^n_{C^\infty(M)}$ for any smooth manifold M. Also since S^1 is 1-dimensional we have $\Omega^1_{S^1}/d\Omega^0_{S^1} = H^1_{DR}(S^1)$. Then our result is

$$HC_n(C^\infty(S^1)) = \begin{cases} C^\infty(S^1) & n \geq 0 \\ 0 & \text{else} \end{cases}$$

Again we can do this for any smooth manifold M.

Example 76. ($\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[z, z^{-1}]$)

Theorem 77. *Let* Γ *be a discrete group. Then*

$$HH_n(\mathbb{C}[\Gamma]) \cong \bigoplus_{\langle\gamma\rangle \in \langle\Gamma\rangle} H_n(K(\Gamma_\gamma, 1); \mathbb{C})$$

Here $\Gamma_\gamma := \{g \in \Gamma | g\gamma = \gamma g\}$ and $K(\pi, n)$ are the Eilenberg-Mac Lane spaces.

Knowing this let us compute $HH_n(\mathbb{C}[\mathbb{Z}])$. First of all $\mathbb{Z}_g = \mathbb{Z}$ for all g since $\mathbb{Z}$ is commutative. Now by algebraic topological methods we know $K(\mathbb{Z}, 1) \cong S^1$. And if we wanted we could beat a dead horse and use the Serre spectral sequence to compute the homology groups of $K(\mathbb{Z}, 1)$, but this is overkill. Instead we can simply state

$$H_n(S^1; \mathbb{C}) = \begin{cases} \mathbb{C} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

So we have

$$HH_n(\mathbb{C}[\mathbb{Z}]) = \begin{cases} \mathbb{C}[\mathbb{Z}] & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

Although there is a small question about why exactly do we know HH_0, HH_1 are $\mathbb{C}[\mathbb{Z}]$, because a priori we only have that they contain $\mathbb{Z}$ copies of $\mathbb{C}$, however since this is a commutative algebra we know $HH_0(\mathbb{C}[\mathbb{Z}]) = \mathbb{C}[\mathbb{Z}]$ and by our theorem it follows that HH_1 must be also.

A quick note of interest: This theorem is related to Shapiro's lemma for group homology.

Now let us compare with $\mathbb{C}[z, z^{-1}]$. This just happens to be a smooth algebra and so we know already

$$HH_n(\mathbb{C}[z, z^{-1}]) = \begin{cases} \mathbb{C}[z, z^{-1}] & n = 0 \\ \mathbb{C}[z, z^{-1}] \cdot dz & n = 1 \\ 0 & \text{else} \end{cases}$$

And because math is nice that way we see that these two results match.

Example 78. (The Weyl Algebra)

Let $A = \mathbb{C}[x, y]/([x, y] - 1)$. We can equivalently view this as $A = \mathbb{C}[d/dt, t]$ or $\mathbb{C}[t, -d/dt]$, but these become more useful when considering module structures. In our case we're viewing A as a $\mathbb{C}$ algebra so we can use the Tor definition of Hochschild homology. Our goal then is to produce a projective resolution that makes our calculation tolerable.

Consider the resolution K_* given by

$$\begin{tikzcd}
 & A \otimes x \otimes A^{op} \arrow[dr] & \\
A \otimes (x \otimes y - y \otimes x) \otimes A^{op} \arrow[ur] \arrow[dr] \arrow[dd, equal] & \oplus & A \otimes A^{op} \arrow[dd, equal] \\
 & A \otimes y \otimes A^{op} \arrow[ur] \arrow[d, equal] & \\
K_2 \arrow[r] & K_1 \arrow[r] & K_0
\end{tikzcd}$$

One can check that this is a subresolution of the bar complex.

We have $HH_n(A) = Tor_n^{A^e}(A, A)$ where $A^e := A \otimes A^{op}$. Now to compute Tor we only need a projective resolution of A as an A^e module. I claim the above resolution is that. Now We have

$$Tor_n^{A^e}(A, A) = H_n(K_* \otimes_{A^e} A) \tag{2.4.2}$$

Our goal at this point is to compute the differentials. Let's notice what happens when we tensor with A^e.

$$\begin{aligned} (b \otimes x \otimes a) \otimes_{A^e} 1 &\quad\leftrightarrow\quad x \otimes (ab) \\ (a \otimes b) \otimes_{A^e} 1 &\quad\leftrightarrow\quad b \cdot 1 \cdot a \end{aligned}$$

Also

$$(1 \otimes x \otimes a) \otimes_{A^e} 1 \xrightarrow{b} (x \otimes a - 1 \otimes xa) \otimes_{A^e} 1 = ax - xa = [a, x] \in A$$

In light of this we can rewrite our resolution $K_* \otimes_{A^e} A$ (abusively) as

$$\begin{tikzcd}
 & x \otimes A \arrow[dr, "{[x,\cdot]}"] & \\
(x \wedge y) \otimes A \arrow[ur, "{[y,\cdot]}"] \arrow[dr, "{-[x,\cdot]}"'] & \oplus & A \\
 & y \otimes A \arrow[ur, "{[y,\cdot]}"'] &
\end{tikzcd}$$

This is not entirely proper, but indeed is the right idea. We will now consider $a \otimes A \cong A$. So if we want we can turn the diagram on it's side and then using the equivalence of $A \cong \mathbb{C}[d/dt, t]$ simultaneously with $\mathbb{C}[t, -d/dt]$ we can consider $[x, a] = \partial a \partial y$ and $[y, a] = -\partial a/\partial x$. Here is makes sense to consider $A = \mathbb{C}[x, y]/(x = \partial/\partial y, y = -\partial/\partial x)$. And then in this convoluted strange scenario we have the following diagram.

$$\begin{tikzcd}
 & \mathbb{C}[x,y] \arrow[dl, "\partial/\partial y"'] \arrow[dr, "\partial/\partial x"] & \\
\mathbb{C}[x,y] \arrow[dr, "-\partial/\partial x"'] & \oplus & \mathbb{C}[x,y] \arrow[dl, "\partial/\partial y"] \\
 & \mathbb{C}[x,y] &
\end{tikzcd}$$

Notice that only one of the $\partial/\partial x$ has a negative sign. This is because we need the total differential to have square 0.

Now we can compute the homology by first computing in the down and left direction, then the down and right direction. This is equivalent to calculating the homology in a spectral sequence. We can see after the first step we have

$$\begin{tikzcd}
 & \mathbb{C}[x] \arrow[dl, "0"'] \arrow[dr, "\partial/\partial x"] & \\
0 \arrow[dr, "-\partial/\partial x"'] & \oplus & \mathbb{C}[x] \arrow[dl, "0"] \\
 & 0 &
\end{tikzcd}$$

Since $ker(\partial/\partial y) = \{\text{functions in } x \text{ alone }\}$. And $Im(\partial/\partial y) = \mathbb{C}[x, y]$ (just integrate with respect to y.)

Our final stage looks like

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dl] \arrow[dr] & \\
0 \arrow[dr] & \oplus & 0 \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

So we've obtained

$$HH_n(\mathbb{C}[x,y]/([x,y]-1)) = \begin{cases} \mathbb{C} & n=2 \\ 0 & \text{else} \end{cases}$$

With this in mind it is VERY easy to compute the cyclic homology using Connes' long exact sequence.

$$\begin{tikzcd}
\cdots \arrow[r] & HH_{2n}(A) \arrow[r] \arrow[d, equal] & HC_{2n}(A) \arrow[r] \arrow[d, equal] & HC_{2n-2}(A) \arrow[r] \arrow[d, equal] & HH_{2n-1}(A) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{C} \arrow[r, "\cong"] & \mathbb{C} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

More simply

$$HC_n(A) = \begin{cases} \mathbb{C} & n \geq 2 \text{ even} \\ 0 & n \text{ odd} \\ 0 & n=0 \end{cases}$$

Example 79. (The Weyl Algebra with formally inverted elements)

Let's see what happens when we formally invert elements from the example above. First, let $A_{(x)} = \mathbb{C}[x,x^{-1},y]/([x,y]-1)$. Notice that the commutation relations can all be derived from $[x,y]=1$.

In this case we can jump directly back to our spectral sequence

$$\begin{tikzcd}
 & \mathbb{C}[x,x^{-1},y] \arrow[dl, "\partial/\partial y"'] \arrow[dr, "\partial/\partial x"] & \\
\mathbb{C}[x,x^{-1},y] \arrow[dr, "-\partial/\partial x"'] & \oplus & \mathbb{C}[x,x^{-1},y] \arrow[dl, "\partial/\partial y"] \\
 & \mathbb{C}[x,x^{-1},y] &
\end{tikzcd}$$

And then again after one stage

$$\begin{tikzcd}
 & \mathbb{C}[x,x^{-1}] \arrow[dl, "0"'] \arrow[dr, "\partial/\partial x"] & \\
0 \arrow[dr, "-\partial/\partial x"'] & \oplus & \mathbb{C}[x,x^{-1}] \arrow[dl, "0"] \\
 & 0 &
\end{tikzcd}$$

Notice what happens at the next stage however. Our top term is simply $ker(\partial/\partial x)$, but our second term is

$$\frac{\mathbb{C}[x, x^{-1}]}{Im(\partial/\partial x)}$$

But $\partial/\partial x$ cannot hit the x^{-1} term, so we're left with

$$\begin{tikzcd}
& \mathbb{C} \arrow[dl] \arrow[dr] & \\
0 \arrow[dr] & \oplus & \mathbb{C} \arrow[dl] \\
& 0 &
\end{tikzcd}$$

Which yields the result

$$HH_n(A_{(x)}) = \begin{cases} \mathbb{C} & n = 2 \\ \mathbb{C} & n = 1 \\ 0 & \text{else} \end{cases}$$

Which yields the result

$$HC_n(A_{(x)}) = \begin{cases} \mathbb{C} & n \geq 1 \\ 0 & n = 0 \end{cases}$$

Now finally let $A_{(x,y)} = \mathbb{C}[x, x^{-1}, y, y^{-1}]/([x, y] - 1)$. Again all the proper commutation relations are determined by $[x, y] = 1$. We have our three stages again, and they change as one would expect.

(1)

$$\begin{tikzcd}
& \mathbb{C}[x, x^{-1}, y, y^{-1}] \arrow[dl, "\partial/\partial y"'] \arrow[dr, "\partial/\partial x"] & \\
\mathbb{C}[x, x^{-1}, y, y^{-1}] \arrow[dr, "-\partial/\partial x"'] & \oplus & \mathbb{C}[x, x^{-1}, y, y^{-1}] \arrow[dl, "\partial/\partial y"] \\
& \mathbb{C}[x, x^{-1}, y, y^{-1}] &
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
& \mathbb{C}[x, x^{-1}] \arrow[dl, "0"'] \arrow[dr, "\partial/\partial x"] & \\
\mathbb{C} \arrow[dr, "-\partial/\partial x"'] & \oplus & \mathbb{C}[x, x^{-1}] \arrow[dl, "0"] \\
& \mathbb{C} &
\end{tikzcd}$$

(3)

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dl] \arrow[dr] & \\
\mathbb{C} \arrow[dr] & \oplus & \mathbb{C} \arrow[dl] \\
 & \mathbb{C} &
\end{tikzcd}$$

Which yields the results

$$HH_n(A_{(x,y)}) = \begin{cases} \mathbb{C} & n = 0, 2 \\ \mathbb{C} \oplus \mathbb{C} & n = 1 \\ 0 & \text{else} \end{cases}$$

$$HC_n(A_{(x,y)}) = \begin{cases} \mathbb{C} & n \geq 0 \text{ even} \\ \mathbb{C} \oplus \mathbb{C} & n \geq 0 \text{ odd} \\ 0 & \text{else} \end{cases}$$

Remark 80. Perhaps being a little abusive with notation, one should realize that the general scheme for computing the Hochschild and cyclic homology is as follows

(1) Write a resolution quasi-isomorphic to the bar resolution. The resolution will look something like this

$$K_* : \cdots \to A \otimes R_n \otimes A^{op} \to A \otimes R_{n-1} \otimes A^{op} \to A \otimes R_{n-2} \otimes A^{op} \to \cdots \tag{2.4.3}$$

Where R_n represents the relations on the algebra in degree n.

(2) Tensor this guy with A over A^e. So we have $K_* \otimes_{A^e} \otimes A$. So we can now compute the Tor groups. The differentials become commutators.

(3) If $HH_*(A)$ is nice enough, then plug it straight into the long exact sequence.

Example 81. (The Noncommutative Torus)

The noncommutative torus shows up all over the place in noncommutative geometry, so it is worth our while to learn a few things about. It will show up again in these notes at least twice.

Let

$$A_\theta = \mathbb{C}[U, V, U^*, V^*]/(UV = qVU)$$

where $q = e^{2\pi i\theta}$ and θ is irrational. It is not necessary to have θ irrational in general, but in that case we'll end up with infinite dimensional HH_* and HC_*. So for now we'll keep θ irrational.

A few quick notes about A_θ before we compute its homologies.

(1) U and V are usually considered unitary operators so that $U^* = U^{-1}$ and $V^* = V^{-1}$

(2) The commutation relations are these $U^k V^l = q^{kl} V^l U^k$

(3) Connes has a classification theorem for projective modules over A_θ. U and V can be realized as unitary operators on $L^2(\mathbb{R})$ and this gives rise to considering $\mathcal{S}(\mathbb{R})$ as a projective module over A_θ. We'll briefly come back to this point in the index theorem on A_θ.

Now we can get to the computation of the homologies. In this case we have our Koszul resolution which looks like

$$\begin{aligned} K_2 = A_\theta \otimes (U \otimes V - qV \otimes U) \otimes A_\theta^{op} &\hookrightarrow A_\theta \otimes A_\theta^{\otimes 2} \otimes A_\theta^{op} \\ K_1 = A_\theta \otimes U \otimes A_\theta^{op} \oplus A_\theta \otimes V \otimes A_\theta^{op} &\hookrightarrow A_\theta \otimes (A_\theta) \otimes A_\theta^{op} \\ K_0 = A_\theta \otimes A_\theta^{op} &= A_\theta \otimes A_\theta^{op}. \end{aligned} \tag{2.4.4}$$

Again we get a diagram

$$\begin{tikzcd} & 0 \arrow[d] & \\ & A_\theta \otimes (U \otimes V - qV \otimes U) \otimes A_\theta^{op} \arrow[dl] \arrow[dr] & \\ A_\theta \otimes U \otimes A_\theta^{op} \arrow[dr] & \oplus & A_\theta \otimes V \otimes A_\theta^{op} \arrow[dl] \\ & A_\theta \otimes A_\theta^{op} & \end{tikzcd}$$

The correct differentials are

$$\begin{tikzcd} left & & right \\ a \otimes (U \otimes V - qV \otimes U) \otimes b \arrow[d] & \leftrightarrow & a \otimes (U \otimes V - qV \otimes U) \otimes b \arrow[d] \\ -qay \otimes x \otimes b + a \otimes x \otimes yb & & ax \otimes y \otimes b - qa \otimes y \otimes xb \\ (a \otimes x \otimes b) \arrow[d] & & (a \otimes y \otimes b) \arrow[d] \\ ax \otimes b - a \otimes xb & & ay \otimes b - a \otimes yb \end{tikzcd}$$

One can check that these give a total differential of zero (i.e. $d^2 = 0$.)

But as we know the situation becomes much simpler when we realize that the differentials are commutators. We get the following diagram.

$$\begin{tikzcd} & A_\theta \arrow[dl, "{[U,\cdot]}"'] \arrow[dr, "{[V,\cdot]}"] & \\ A_\theta \arrow[dr, "{-[V,\cdot]}"'] & \oplus & A_\theta \arrow[dl, "{[U,\cdot]}"] \\ & A_\theta & \end{tikzcd}$$

Our job is to compute kernels of commutators. First we have

$$[U, U^k V^l] = U^{k+1} V^l - q^l U^{k+1} V^l = (1 - q^l) U^{k+1} V^l.$$

This tells us that the only kernel is when $q^l = 1$, but we've carefully avoided this except when $l = 0$ since we've mandated that θ be irrational.
Similarly

$$[V, U^k V^l] = q^{-k} U^k V^{l+1} - U^k V^{l+1} = (q^{-k} - 1) U^k V^{l+1}$$

Again the kernel is only when $k = 0$. If we allowed q to be a root of unity then we would have infinite dimensional kernel. As it is we can compute just as before, and the calculation is similar to that of $A_{(x,y)}$ in the previous example. We're left with

$$HH_n(A_\theta) = \begin{cases} \mathbb{C} & n = 0, 2 \\ \mathbb{C} \oplus \mathbb{C} & n = 1 \\ 0 & \text{else} \end{cases}$$

And this leaves us with the same thing as above

$$HC_n(A_\theta) = \begin{cases} \mathbb{C} & n \geq 0 \text{ even} \\ \mathbb{C} \oplus \mathbb{C} & n \geq 0 \text{ odd} \\ 0 & \text{else} \end{cases}$$

Example 82. A Kunneth theorem example

Let's begin with the Kunneth theorems in several guises.

(1) For homology (singular, simplicial, CW, etc.) we have for flat R-modules C_n and $B_n(C)$ the following sequence is exact.

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to H_n(C \otimes C') \to \bigoplus_{p+q=n-1} Tor_1^R(H_p(C), H_q(C')) \to 0 \tag{2.4.5}$$

(2) For Hochschild homology we have the same thing. But if the algebras A and $HH_*(A)$ are flat over $\mathbb{C}$ then

$$HH_*(A) \otimes HH_*(A') \cong HH_*(A \otimes A') \tag{2.4.6}$$

(3) For cyclic homology as stated above

$$\cdots \to HC_n(C \otimes C') \xrightarrow{i} \bigoplus_{p+q=n} HC_p(C) \otimes HC_q(C') \overset{S\otimes 1 - 1 \otimes S}{\longrightarrow}$$

$$\bigoplus_{p+q=n-2} HC_p(C) \otimes HC_q(C') \xrightarrow{\partial} HC_{n-1}(C \otimes C') \to \cdots \tag{2.4.7}$$

With these wonderful results in mind let's look at a few examples. First, let $\mathcal{A} = W[t]$ where W is the Weyl algebra. That is $\mathcal{A} = \mathbb{C}[x, y, t]/([x, y] - 1)$. This is equivalent (algebraically) to

$$\mathcal{A} = \mathbb{C}[x, y]/([x, y] - 1) \otimes \mathbb{C}[t]$$

Notice that $\mathbb{C}[t]$ and $HH_*(\mathbb{C}[t])$ are flat over $\mathbb{C}$ so we can apply our theorems. We get

$$\begin{aligned} HH_3(\mathcal{A}) = HH_3(W) \otimes HH_0(\mathbb{C}[t]) \oplus HH_2(W) \otimes HH_1(\mathbb{C}[t]) \\ \oplus HH_1(W) \otimes HH_2(\mathbb{C}[t]) \oplus HH_0(W) \otimes HH_3(\mathbb{C}[t]) \\ = \mathbb{C} \otimes \mathbb{C}[t] \oplus 0 = \mathbb{C}[t] \end{aligned}$$

and

$$\begin{aligned} HH_2(\mathcal{A}) = HH_2(W) \otimes HH_0(\mathbb{C}[t]) \oplus HH_1(W) \otimes HH_1(\mathbb{C}[t]) \\ \oplus HH_0(W) \otimes HH_2(\mathbb{C}[t]) = \mathbb{C}[t] \end{aligned}$$

and

$$HH_1(\mathcal{A}) = HH_1(W) \otimes HH_0(\mathbb{C}[t]) \oplus HH_0(W) \otimes HH_1(\mathbb{C}[t]) = 0$$

and

$$HH_0(\mathcal{A}) = HH_3(W) \otimes HH_0(\mathbb{C}[t]) = 0$$

A more concise (and easier on the eyes version of this is

$$HH_n(\mathcal{A}) = \begin{cases} \mathbb{C}[t] & n = 2, 3 \\ 0 & \text{else} \end{cases}$$

Now we have a somewhat more arduous task; compute the cyclic homology of $\mathcal{A}$. In order to do this let's first recall the cyclic homology of our two basic algebras.

The Weyl algebra

$$HC_n(A) = \begin{cases} \mathbb{C} & n \geq 2 \text{ even} \\ 0 & n \text{ odd} \\ 0 & n = 0 \end{cases}$$

And $\mathbb{C}[t]$

$$HC_n(A) = \begin{cases} \mathbb{C}[t] & n \text{ even} \\ \mathbb{C}[t] & n \text{ odd} \end{cases}$$

Now we appeal to the long exact sequence relating our cyclic homologies. One can spend a long time writing out terms, or I can just give the answer. In this case it is

$$HC_n(\mathcal{A}) = \begin{cases} \mathbb{C}[t] & n \geq 3 \\ 0 & \mathit{else} \end{cases}$$

Truly it is easier to compute this the normal way using Connes' exact sequence relating HH_* and HC_*, but it certainly works to compute it using this Kunneth type formula.

Notice that any of our algebras above (Weyl, Weyl with formal inverses, NCT, polynomial algebras) have the required properties of flatness. Therefore we can tensor any of them with zeal and essentially read their homologies right off our previous results.

Example 83. ($\mathbb{C}_q[SU(2)]$ and $\mathbb{C}_q[SL(2)]$) I don't actually intend to compute anything here. I'm basically going to read the results right of a paper of Feng and Tsygan and plug in the example of $\mathbb{C}_q[SL(2)]$. The main theorem from Feng and Tsygan is this

Theorem 84. *The Hochschild homology of a quantum algebraic group is*

$$HH_n(\mathbb{C}_\hbar[G]) \cong \tilde{\Omega}_n[N_G] \otimes \mathbb{C}((\hbar)) \tag{2.4.8}$$

Where we have the following definitions

1. N_G = normalizer of the Cartan subgroup H in $K_{\mathbb{C}} \cong G$. Here, we're assuming G is a complex Lie group.
2. $\tilde{\Omega}_n[N_G] = \tilde{\mathbb{C}}[N_G] \otimes \Lambda^n \mathbf{a}_{\mathbb{C}}$
3. $\tilde{\mathbb{C}}[N_G] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}^{l(\lambda)}$ Here Λ is a weight lattice for the Weyl group.
4. $\mathbf{a}_{\mathbb{C}} = \{(x, -x) : x \in \mathbf{h}\}$ is from the Iwasawa decomposition of $\mathbf{g}_{\mathbb{C}} = \mathbf{k}_{\mathbb{C}} \oplus \mathbf{a}_{\mathbb{C}} \oplus \mathbf{n}_{\mathbb{C}}$.

In the case of $\mathbb{C}_q[SL(2)]$ Feng and Tsygan have computed

$$HH_n(\mathbb{C}_q[SL(2)]) \cong H_n(\mathbf{g}^*, \mathbb{C}[SL(2)] \otimes \mathbb{C}((q)))$$

and

$$HC_n(\mathbb{C}_q[SL(2)]) \cong \begin{cases} H_0(\mathbf{g}^*, \mathbb{C}[SL(2)] \otimes \mathbb{C}((q))) & n = 0 \\ H^1_{DR}(T_{\mathbb{C}}) \otimes \mathbb{C}((q)) & n \text{ odd} \\ H^0_{DR}(T_{\mathbb{C}}) \otimes \mathbb{C}((q)) & n \geq 2 \text{ even} \end{cases}$$

In the case of $\mathbb{C}_q[SU(2)]$ we realize that $SU(2)$ is a real form of $SL(2)$ And we have a somewhat odd computation to make. If $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ then we'll get

$$HH_n(A_{\mathbb{C}}) = HH_n(A) \otimes_{\mathbb{R}} \mathbb{C}$$

So we see that all real forms of a complex Lie group may produce the same homology. At least we have that they are isomorphic after tensoring with $\mathbb{C}$.

In their paper *Noncommutative Differential Geometry of Quantum SU(2)*, Masuda, Nakagami, and Watanabe construct an explicit resolution and give the corresponding differentials.

3. A Few Index Theorems

In this section I intend to present a few examples and basic definitions associated with index theory. In particular the proof of [ENN] will be sketched. An index theorem by Connes on the noncommutative torus will also be sketched.

3.1. **Very Basic Index Theory.** There are many equivalent formulations of the index of the operator. The most basic of these are

$$index(P) = dim(ker(P)) - dim(coker(P)) \tag{3.1.1}$$

$$index(P) = dim(ker(P)) - dim(ker(P^*)). \tag{3.1.2}$$

Of course we usually require that both of these numbers be finite in order that this makes sense. We'll impose the condition that $P : V_1 \to V_2$ be a map between vector spaces (Banach,Hilbert,etc.) With finite dimensional kernel and cokernel. But before I begin cheating I should note that these finite dimensional constraints are not always considered. Such operators are called *Fredholm*. For the purposes of these notes I will consider Fredholm operators unless otherwise explicitly noted. It may also be useful for the reader to read these preliminary notes with an eye toward applications of ΨDOs.

Let us take a quick look at why the two stated formulations of the index are equivalent.

Lemma 85. *Let $M : X \to U$ be Fredholm. Then X has a unique decomposition $X = ker(M) \oplus Y_1$ where Y_1 is the orthocomplement of $ker(M)$. Moreover U has a similar decomposition $U = Range(M) \oplus V$. $M^* : U \to X$ yields similar decompositions. And*

$$dim(coker(M)) = dim(ker(M^*))$$

Proof. Y is isomorphic to X/N_M where $N_M := ker(M)$. We then have that $M : Y \to Range(M)$ is invertible. But we then have $V \cong U/Range(M)$. But by similar decompositions from M^* we see $U = N_{M^*} \oplus Y_2$. Now we need $dim(N_{M^*}) = dim(U/R_M)$. It is easy to see however that $R_{M^*} \subset N_M$ by considering the pairing $0 = \langle Mx, y\rangle = \langle x, M^*y\rangle$. The same holds when we interchange the roles of M and M^* so we get our result. ☐

Theorem 86. *An operator P has finite index if and only if P has a pseudoinverse (or parametrix for ΨDO). By this we mean $P : V_1 \to V_2$ and $P' : V_2 \to V_1$ such that*

$$PP' = I_{V_2} + K_2, \quad P'P = I_{V_1} + K_1$$

where K_1 and K_2 are degenerate (compact/finite rank) operators on V_1 and respectively V_2.

Theorem 87. *Let $P_1 : V_1 \to V_2$ and $P_2 : V_2 \to V_3$ then*

$$index(P_1P_2) = index(P_1) + index(P_2)$$

Lemma 88. *If K is a degenerate (compact/finite rank) operator and P has finte index then*

$$index(P + K) = index(P).$$

Proof. Since P has finite index it has a pseudoinverse Q. That is $PQ = I + K$ for some compact K. Now for compact L we have

$$(P + L)Q = PQ + LQ = I + K + LQ$$

but since L is compact so is LQ. Then Q is a pseudoinverse for $P + L$ too. We have

$$index(P + L) = -index(Q) = index(P).$$

□

Corollary 89. *If P and Q are pseudoinverses then $index(P) = -index(Q)$.*

It is important to note (if not by now obvious) that $index(I) = 0$ and also $index(I + K) = 0$ for any compact K. Now we are ready to state some very important properties of the index.

Theorem 90. *Let $P_n : U \to V$ and $Q_n : V \to U$ sequences of linear maps which converge strongly and are pseudoinverses for each n. Then*

$$\lim_{n\to\infty} P_n = P$$

Theorem 91. *(Homotopy invariance of index) Let $P(t) : U \to V$ and $Q(t) : V \to U$ be contiuous with respect to $0 \le t \le 1$ and pseudoinverses for each t then $index(P(t)) = index(P(0))$ $\forall t \in [0, 1]$.*

One of the main problems that arises in index theory is that such a simple definition will certainly lead to computational difficulty. When we're simply dealing with matrices calculating kernels and cokernels is a matter of linear algebra. But when we move to a function space or $L^2(M)$ calculating dimensions of kernels becomes a somewhat arduous task. Therefore many great mathematicians have devoted great amounts of time to finding explicit formulae for indices. I hope to present at least two in the following sections. Now let us look at a quick example.

Example 92. Consider the operator $a = x + d/dx$ defined on Schwartz functions or by extension $L^2(\mathbb{R})$. We can directly compute that $index(a) = 1$ by solving simple differential equations. Also a plays an important role in quantum mechanics as the *annihilator* or *lowering operator* for the quantum harmonic oscillator. Of course physicists will tell you that it is missing a phase factor, but fortunately multiplying by constants does not change the index. In fact if we wanted to be truly pedantic about it we could calculate $index(c) = 0$ for any $c \in \mathbb{C}$ and then say $index(ca) = index(c) + index(a) = index(a)$. But this is unnecessary.

Let us calculate the dimension of the kernel of a.

$$\begin{aligned}
(x + d/dx)f &= 0 \\
df/dx &= -xf \\
\int (1/f)(df/dx)dx &= \int -xdx \\
ln(f) &= -x^2/2 + c \\
f &= Ce^{-x^2/2}
\end{aligned}$$

Here $c \in \mathbb{R}$ and $C = e^c$ are just constants. We see then that $ker(a)$ is one dimensional spanned by $e^{-x^2/2}$ which you might recognize as the ground state of the quantum harmonic oscillator.

Now let us calculate $coker(a)$. We can also solve this by first order ODE. We need to check if we can solve

$$xf + df/dx = g$$

for any $g \in \mathcal{S}(\mathbb{R})$.
Recognizing that our integrating factor is $e^{x^2/2}$ and we get

$$f = e^{-x^2/2} \int g e^{x^2/2} dx$$

So we can (in principle) solve this ODE for any given $g \in \mathcal{S}(\mathbb{R})$ hence the range is everything and $coker(a) = 0$. Then

$$index(a) = dim(ker(a)) - dim(coker(a)) = 1 - 0 = 1.$$

After solving this ODE explicitly we may realize that our operator a may be factored as $e^{-x^2/2}(d/dx)(e^{x^2/2}\bullet)$ i.e. $af = e^{-x^2/2}(d/dx)(e^{x^2/2}f)$ and then it is easy to seethe kernel and cokernel.

One other interesting thing about this example comes from the result $index(PQ) = index(P) + index(Q)$ and then applying this to

$$index(a^n) = n \cdot index(a) = n$$

we see that the kernel is spanned for the n least excited states of the oscillator.

3.2. **The Algebraic Index Theorem.** This section is devoted to one of the main ideas in the [ENN] proof. These notes are based on private discussions with B. Tsygan. The algebraic index theorem may be stated as follows.

We begin with the following assumptions. Let $H_\pm$ be Hilbert spaces and $D_\pm$ operators such that $D_+ : H_+ \to H_-$ and $D_- : H_- \to H_+$ with $1 - D_- D_+, 1 - D_+ D_-$ trace class and $\mathcal{A} : H_\pm \to H_\pm$ and algebra of operators such that $[D_\pm, a]$ is trace class $\forall a \in \mathcal{A}$. Finally, let $e \in \mathcal{A}$ be an idempotent.

Theorem 93. *We have a formula of the type* $index(eD_+e) = \langle cocycle_{D_\pm}, cycle_e \rangle$. *Furthermore the cocycle and cycle are given by the following:*
$cycle_e = ch([e]) = e + \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{n!}(e - \frac{1}{2}) \otimes e^{\otimes 2n} \in CC_0^-(\mathcal{A})$
$cocycle_{D_\pm}(a_0) = tr(a_0[D_+, D_-]) \in CC_-^0(\mathcal{A})$

The discourse that follows will not be a formal proof, but more like a discussion on why the theorem is true and some derivation. One must pardon me a little if these notes seem clunky and disconnected.
Let A be an algebra on which H is represented. Let $\mathfrak{g} = Der(A)$, and $A[1]$ the $\mathfrak{g}$ module A with degree shifted by 1. Then we can define the following map

$$i_D : A \otimes A^{\otimes n} \to A \otimes A^{\otimes (n - \frac{1}{2}) \pm \frac{1}{2}}$$

given by

$$i_D(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 D(a_1) \otimes a_2 \otimes \cdots \otimes a_n & D \in \mathfrak{g} \\ a_0 D \otimes a_1 \otimes \cdots \otimes a_n & D \in A \end{cases}$$

Now let $D_j \in \mathfrak{g} \subset Der(A)$ for each j then we can define the map

$$\tau_{D_1 \wedge \cdots \wedge D_n}(a_0 \otimes \cdots \otimes a_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma (i_{D_{\sigma(1)}} \cdots i_{D_{\sigma(n)}}(a_0 \otimes \cdots \otimes a_n))$$

Here it should be noted that we are working toward building an equivalent double complex to that of the (b, B) bicomplex and then using a pairing of the new cycles and cocycles to achieve our index theorem. With that in mind we forge ahead with still more definitions and assumptions.
Let δ be a derivation such that

$$\delta : \begin{cases} \mathfrak{g} \to 0 \\ A[1] \to \mathfrak{g} \end{cases}$$

Which is given by

$$\delta(x) = [x, \cdot] \ \ \forall x \in A.$$

With all this in mind we have two very nice consequences

1. $\tau_{D_1 \wedge \cdots \wedge D_n} \circ b = \sum_{i=1}^{n} \pm \tau_{D_1 \wedge \cdots \wedge \delta D_i \wedge \cdots \wedge D_n}$
2. $\tau_{D_1 \wedge \cdots \wedge D_n} \circ B = \sum_{i<j} \pm \tau_{[D_i, D_j] \wedge D_1 \wedge \cdots \wedge \hat{D}_i \wedge \cdots \wedge \hat{D}_j \wedge \cdots \wedge D_n}$

Now let us assume $D_{[+,-]} := [D_+, D_-]$ has finite rank. It is convenient for intuition to think of these operators as commuting. For now we'll consider $D_[+,-] = 0$. Let's look at a new cocycle for $D_\pm$

$$\begin{aligned} \mathit{cocycle}_{D_\pm}(a_0, a_1, a_2) &= \frac{1}{2} tr(a_0[D_+, a_1][D_-, a_2] - a_0[D_-, a_1][D_+, a_2]) \\ &= \tau_{[D_+, \bullet] \wedge [D_-, \bullet]}(a_0, a_1, a_2) \end{aligned}$$

Finally let's define our other derivation which we'll call ∂_{Lie}.

$$\partial_{Lie}(D_1 \wedge \cdots \wedge D_n) = \sum_{i<j} [D_i, D_j] \wedge D_1 \wedge \cdots \wedge \hat{D}_i \wedge \cdots \wedge \hat{D}_j \wedge \cdots \wedge D_n.$$

With our defintions of δ and ∂_{Lie} we have the intertwining of the two following diagrams.

(1)

$$\begin{tikzcd} & {[D_+, \bullet] \wedge [D_-, \bullet]} \arrow[r, mapsto, "\delta"] \arrow[d, "\partial_{Lie}"] & 0 \\ D_{[+,-]} \arrow[r, mapsto, "\delta"] & {[D_{[+,-]}, \bullet]} & \end{tikzcd}$$

(2)

$$\begin{tikzcd} & \tau_{[D_+, \bullet] \wedge [D_-, \bullet]} \arrow[r, mapsto, "b"] \arrow[d, "B"] & \tau_{\delta(\bullet)} = 0 \\ \tau_{D_{[+,-]}} \arrow[r, mapsto, "b"] & \tau_{[[D_-, D_+], \bullet]} = \tau_{\delta(D_{[+,-]})} & \end{tikzcd}$$

Remark 94. There are several important comments to be made here.

1. In the top right corner we have $\delta(\bullet) = 0$. This follows from the fact that $\bullet \in \mathfrak{g}$ and from the fact that
$$\delta : \begin{cases} \mathfrak{g} \to 0 \\ A[1] \to \mathfrak{g} \end{cases}$$
2. The right hand term on the bottom row has a similar situation except that $[D_+, D_-] \in A$ hence $\delta([D_+, D_-]) \in \mathfrak{g}$

(3) What we have shown is that for $c \in \Lambda^*(\mathfrak{g} + A[1])$ the following formulas hold

$$\tau_c \circ b = \tau_{\delta(c)} \tag{3.2.1}$$

$$\tau_c \circ B = \tau_{\partial_{Lie}(c)} \tag{3.2.2}$$

At this point we're almost ready to show our pairing explicitly. Let us denote

$$\tau_0(a_0) = cocycle_{D_\pm}(a_0) = tr(a_0[D_+, D_-]) \tag{3.2.3}$$

$$and \;\; \tau_2(a_0, a_1, a_2) = cocycle_{D_\pm}(a_0, a_1, a_2) =$$

$$\frac{1}{2} tr(a_0[D_+, a_1][D_-, a_2] - a_0[D_-, a_1][D_+, a_2]) \tag{3.2.4}$$

This is rather unfortunate notation in reusing τ, but this is what is commonly used for cocycles. Notice that we have

$$\begin{tikzcd}
A \otimes \bar{A} \arrow[r, "B"] \arrow[d, "b"] & A \otimes \bar{A} \otimes \bar{A} \arrow[d, "\tau_2"] \\
A \arrow[r, "\tau_0"] & \mathbb{C}
\end{tikzcd}$$

where

$$\tau_2 \circ B + \tau_0 \circ b = 0$$

hence we have intertwined our new complex with the (b, B) bicomplex. Moreover this diagram is a part of the diagram for CC_0^-

$$\begin{tikzcd}
 & & \arrow[r, "B"] \arrow[d, "b"'] & A^{\otimes 7} \\
 & \arrow[r, "B"] \arrow[d, "b"'] & A^{\otimes 5} & \\
\arrow[r, "B"] \arrow[d, "b"'] & A^{\otimes 3} \arrow[d, "\tau_2"'] & & \\
A \arrow[r, "\tau_0"] & \mathbb{C} & &
\end{tikzcd}$$

Here we were looking for a cocyle that annihilates the image of a differential so as to make it a cocycle in $CC(A)$ and hence here in CC_0^- as shown.

Finally we have

$$index(eD_+e) = Tr(eD_+e \cdot eD_-e - eD_-e \cdot eD_+e) \tag{3.2.5}$$

But we've assumed $[D_+, D_-]$ and $[D_\pm, e]$ are trace class so by so algebraic manipulations one will arrive at

$$ind(eD_+e) = tr((e - \frac{1}{2})[D_+, e][D_-, e] - (e - \frac{1}{2})[D_-, e][D_+, e])$$

$$= \tau_2(e - \frac{1}{2}, e, e) = \langle cocycle_{D_\pm}, cycle_e \rangle \tag{3.2.6}$$

A slightly more specific index formula of the same type for elliptic (pseudo)differential operators is constructed in a somewhat different and more specific way.

3.3. **The Index Theorem of E-N-N.** In 1995 [ENN] gave a new approach to the classical Atiyah-Singer index theorem. They used the technique of continuous fields of C^*-algebras. The original proof in 1995 is only for elliptic ΨDO on $\mathbb{R}^n$. There are also a few small differences in the classes of ΨDO and symbols used in [ENN] as opposed to those of Melrose or Shubin.

Without prolonging it too much let's state the theorem outright and then do some work and try to explain why it is true.

Theorem 95. *Let P_a be and elliptic ΨDO on $\mathbb{R}^n$ defined in the sense of [ENN] (which is stronger than that of Melrose and Shubin) then*

$$index(P_a) = \frac{1}{(2\pi i)^n n!} \int_{T^*\mathbb{R}^n} tr(\hat{e}_a (d\hat{e}_a)^{2n}) \tag{3.3.1}$$

Where $T^*\mathbb{R}^n$ is oriented by $dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_n \wedge d\xi_n$ and

$$e_a = \begin{pmatrix} (1+a^*a)^{-1} & (1+a^*a)^{-1}a^* \\ a(1+a^*a)^{-1} & a(1+a^*a)^{-1}a^* \end{pmatrix} \tag{3.3.2}$$

and

$$\hat{e}_a = e_a - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.3.3}$$

Now that we know the statement of the theorem let's make sure we have all our definitions down. An important difference to note is the following

Definition 96. An M_k valued-symbol a of order m is said to be *elliptic* if $\exists C, R > 0$ such that

$$a(x,\xi)^* a(x,\xi) \geq C(|x|^2 + |\xi|^2)^m I_k \ \ |x|^2 + |\xi|^2 \geq R$$

Notice that the difference here is in the growth bound on the x variable as well rather than just the ξ variable. Another subtlety to notice is that [ENN] use the left quantization exclusively. Of course we know this doesn't make terribly much difference but we have

$$\begin{aligned} (P_a u)(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\langle x,\xi\rangle} a(x,\xi)(\mathcal{F}u)(\xi) d\xi \\ &= (2\pi)^{-n} \int_{T^*\mathbb{R}^n} e^{\langle x-y,\xi\rangle} a(x,\xi) u(y) dy d\xi \end{aligned} \tag{3.3.4}$$

Example 97. Before we embark on the proof of the theorem, let's revisit an example we know well. Let $a(x,\xi) = x + i\xi$ then $P_a = x + d/dx$. We know from a previous example that $index(P_a) = 1$. Now let's use the theorem to compute it. Do not be fooled, even though the theorem is stated in a concise formula, these integrals are fairly horrible. In this case (the easy nontrivial case) we have

$$index(P_a) = \frac{1}{2\pi i} \int_{\mathbb{R}_x, \mathbb{R}_\xi} tr\left(\begin{pmatrix} \frac{1}{1+x^2+\xi^2} & \frac{x-i\xi}{1+x^2+\xi^2} \\ \frac{x+i\xi}{1+x^2+\xi^2} & \frac{-1}{1+x^2+\xi^2} \end{pmatrix} d \begin{pmatrix} \frac{1}{1+x^2+\xi^2} & \frac{x-i\xi}{1+x^2+\xi^2} \\ \frac{x+i\xi}{1+x^2+\xi^2} & \frac{-1}{1+x^2+\xi^2} \end{pmatrix}^2 \right) \tag{3.3.5}$$

And of course since we're working with $T^*\mathbb{R}$ we have

$$da(x,\xi) = \frac{\partial a(x,\xi)}{\partial x} dx + \frac{\partial a(x,\xi)}{\partial \xi} d\xi$$

It should be noted further that multiplying

$$\begin{aligned}(a_1dx + b_1d\xi)\cdot(a_2dx + b_2d\xi) &= (a_1a_2dx\wedge dx + a_1b_2dx\wedge d\xi + b_1a_2d\xi\wedge dx + a_2b_2d\xi\wedge d\xi)\\ &= (a_1b_2 - b_1a_2)dx\wedge d\xi\end{aligned}$$

To make a long story short, one has to take a lot of derivatives and be careful with the order of multiplying matrices and carrying complex numbers. Once all that is said and done one will arrive at

$$tr(\hat{e}_a(d\hat{e}_a)^2 = \frac{2idx\wedge d\xi}{(1+x^2+\xi^2)^3})$$

Then our integral reduces to

$$\frac{1}{\pi}\int_{T*\mathbb{R}}\frac{dx\wedge d\xi}{(1+x^2+\xi^2)^3}$$

then one can convert to polar coordinates and arrive at

$$\frac{1}{\pi}\int_0^\infty\int_0^{2\pi}\frac{rdrd\theta}{(1+r^2)^3}$$

And from there is an "easy" integral which we evaluate to be 1. Hence

$$index(x + d/dx) = \frac{1}{2\pi i}\int tr(\hat{e}_a(d\hat{e}_a)^2) = 1$$

Which is indeed the result we wanted since we already computed this index explicitly in §3.1.

We'll now jump to the proof of the main theorem and I'll give interludes to fill in as many missing details as I can.

The proof of [ENN] begins with proving the continuity of the field of graph projections. Given an elliptic operator $P : \mathcal{S}(\mathbb{R}^n; V) \to \mathcal{S}(\mathbb{R}^n; V)$ and its graph projection e (e_a above) we have that $e \in M_2(\mathcal{K}(L^2(\mathbb{R}^n; V))^\sim)$ where V is some finite dimensional $\mathbb{C}$ vector space and $A^\sim$ denotes an algebra with unit adjoined. If $dimV = d$ then

$$\mathcal{K}(L^2(\mathbb{R}^n; V)) = M_d(\mathcal{K}(L^2(\mathbb{R}^n)))$$

and similarly with units adjoined

$$\mathcal{K}(L^2(\mathbb{R}^n; V)^\sim) = M_d(\mathcal{K}(L^2(\mathbb{R}^n)))^\sim \subseteq M_d(\mathcal{K}(L^2(\mathbb{R}^n))^\sim)$$

So we have

$$\hat{e} = e - \begin{pmatrix} 0 & 0 \\ 0 & I_k \end{pmatrix} \in M_{2d}(\mathcal{K}(L^2(\mathbb{R}^n)))$$

From here the outline of the proof looks like this

(1) Realize by a technical lemma that we have a continuous field of C^* algebras given by $\hbar \mapsto e_\hbar^\infty$ which is continuous $\forall \hbar \in [0, 1]$

(2) Define a cocycle ω acting on the subspace $\mathcal{K}^\infty$ of integral operators with kernels in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$

(3) Define a cocycle ε_0 on $\mathcal{S}(T^*\mathbb{R}^n)$

(4) Recall exactly what the hell we mean by $\rho_\hbar$ and $\hat{f}_k$

(5) Show $\lim_{\hbar\to 0}\omega(\rho_\hbar(\hat{f}_0),\dots,\rho_\hbar(\hat{f}_{2n})) = \varepsilon_0(\rho_0(\hat{f}_0),\dots,\rho_0(\hat{f}_{2n}))$

(6) Show

$$\begin{aligned} &\lim_{\hbar\to 0}(\omega\sharp tr)(\rho_\hbar(\hat{f}_0),\dots,\rho_\hbar(\hat{f}_{2n})) \\ &= \frac{1}{(2\pi i)^n n!}\int_{T^*\mathbb{R}^n\times\{0\}} tr(f_0 df_1\wedge\cdots\wedge df_{2n}) \end{aligned}$$

(7) Appeal to homotopy invariance of index in the guise of $index(P) = \langle\omega_\hbar, e_\hbar\rangle = \langle\omega_\hbar, e_\hbar\rangle|_{\hbar=0}$

(8) Recognize we have a string of equalities

$$\begin{aligned} index(P) &= \langle[e_1]-[\begin{pmatrix}0&0\\0&1\end{pmatrix}],[(2\pi i)^n n!\omega_\hbar]\rangle \\ &= \langle[e_1^\infty]-[\begin{pmatrix}0&0\\0&1\end{pmatrix}],[(2\pi i)^n n!\omega_\hbar]\rangle \\ &= \langle[e_\hbar^\infty]-[\begin{pmatrix}0&0\\0&1\end{pmatrix}],[(2\pi i)^n n!\omega_\hbar]\rangle \\ &= \langle[e_0^\infty]-[\begin{pmatrix}0&0\\0&1\end{pmatrix}],[(2\pi i)^n n!\varepsilon_0]\rangle \\ &= \langle[e_a]-[\begin{pmatrix}0&0\\0&1\end{pmatrix}],[(2\pi i)^n n!\varepsilon_0]\rangle \\ &= \frac{1}{(2\pi i)^n n!}\int tr(\hat{e}_a(d\hat{e}_a)^{2n}) \end{aligned}$$

(9) A small K-theoretic argument that $[e_a]-[\begin{pmatrix}0&0\\0&1\end{pmatrix}] = [e_0^\infty]-[\begin{pmatrix}0&0\\0&1\end{pmatrix}]$ and that this indeed is the same as $[Ker(P)]-[Ker(P^*)]$

Our first point is the product of much hard work. I refer the reader to the paper itself.

The second point is the definition of ω. This requires a little work. Our subalgebra $\mathcal{K}^\infty$ is identified with the *-subalgebra $\{\rho_\hbar(\hat{f}); f\in\mathcal{S}(T^*\mathbb{R}^n\times\mathbb{R})\}$ when $\hbar>0$. When $\hbar=0$ we simply have $\{\rho_0(\hat{f}); f\in\mathcal{S}(T^*\mathbb{R}^n\times\mathbb{R})\}$ which is identified with $\mathcal{S}(T^*\mathbb{R}^n)\subset C_0(T^*\mathbb{R}^n)$. It is then noted that these *-subalgebras are stable under holomorphic functional calculus.

Continuing [ENN] say that it is a "well known" result that $\mathcal{K}^\infty\subseteq\mathcal{L}_1(L^2(\mathbb{R}^n))$. So we can denote by Tr the restriction of the standard trace from $\mathcal{K}(L^2(\mathbb{R}^n))$ which is to say summing the eigenvalues like any good trace does on trace class operators. Now we go through a slew of definitions.

Definition 98. For $j=1,\dots,n$ we make the following definitions

(1) $D_j = \partial/\partial x_j$

(2) $M_j =$ multiplication by x_j

(3) $\delta_{2j-1}(T) = [D_j, T]$

(4) $\delta_{2j}(T) = [M_j, T]$

Where $T \in \mathcal{K}^\infty$.

These δ_j have important properties. Since $T \in \mathcal{K}^\infty$ we get that $[D_j, T], [M_j, T]$ extend to compact operators. That is to say $\delta_j : \mathcal{K}^\infty \to \mathcal{K}^\infty$ is a derivation for all j. Furthermore $\delta_j \delta_k = \delta_k \delta_j$. And $Tr(\delta_j(T)) = 0$ for any $T \in \mathcal{K}^\infty$ (since Tr kills commutators.) Finally we come to our definition of ω. Let $T_j \in \mathcal{K}^\infty$ then a $2n+1$ linear functional is given by

$$\omega(T_0, \ldots, T_{2n}) = \frac{(-1)^n}{n!} \sum_{\sigma \in S_{2n}} sgn(\sigma) Tr(T_0 \delta_{\sigma(1)}(T_1) \cdots \delta_{\sigma(2n)}(T_{2n})) \tag{3.3.6}$$

We need to verify that ω is indeed a $2n$-cocycle. In order to verify this we only need to check two things.

(1) $(1-t)\omega = 0$ where t is the cyclic operator

(2) $b\omega(T_0, T_1, \ldots, T_{2n+1}) = 0$

Let us quickly check these things.

(1) $(1-t)\omega(T_0, \ldots, T_{2n}) = \omega(T_0, \ldots, T_{2n}) - (-1)^{2n}\omega(T_{2n}, T_0, \ldots, T_{2n-1})$
We can see this equivalence just by using the definition of ω

(2) $b(\omega)(T_0, \ldots, T_{2n+1}) = \sum_{i=0}^{2n} \omega(T_0, \ldots, T_i T_{i+1}, \ldots, T_{2n+1}) + (-1)^{2n+1}\omega(T_{2n+1}T_0, T_1, \ldots, T_{2n})$
This requires a little more subtlety. Let's examine the general i^{th} term. We don't need $i = 0$ or $i = 2n+1$ for the time being. One piece of the sum is $\frac{(-1)^n}{n!} \sum_{\sigma \in S_{2n}} \omega(T_0, \delta_{\sigma(1)}(T_1) \ldots \delta_{\sigma(i)}(T_i T_{i+1}) \delta_{\sigma(i+1)}(T_{i+2}) \ldots \delta_{\sigma(2n)}(T_{2n+1}))$
But since each δ_j is a derivation we have $\delta_j(T_i T_k) = T_i \delta_j(T_k) + \delta_j(T_i) T_k$. Furthermore ω is a linear functional, so we have a monster expansion of terms. They will all appear twice, but once with a $+$ and once with a $-$ and therefore they all cancel so we get that ω is a cocycle.

The third point is simply given by defining ε_0

$$\varepsilon_0(f^0, \ldots, f^{2n}) = \int_{T^*\mathbb{R}^n} f^0 df^1 \wedge \cdots \wedge df^{2n} \tag{3.3.7}$$

Here, $f^i \in \mathcal{S}(T^*\mathbb{R}^n)$ and $T^*\mathbb{R}^n$ is oriented as before by $dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_n \wedge d\xi_n$.

Our fourth point is not terribly exciting, but it is important. Let $f \in \mathcal{S}(T^*\mathbb{R}^n \times \mathbb{R})$ then we make the following definitions

Definition 99.

$$\hat{f}(t, x, \hbar) = \int_{\mathbb{R}^n} f(t, \xi, \hbar) e^{-i\langle x, \xi \rangle} d\xi$$

and then we define

$$\rho_\hbar(\hat{f})\phi(t) = \frac{1}{(2\pi)^n} \int \hat{f}(x, y, \hbar)\phi(x + \hbar y) dy$$

Now $\rho_\hbar(\hat{f})$ is a compact operator on $L^2(\mathbb{R}^n)$. We should also note that in the case $\hbar = 0$ we have $\rho_0(\hat{f}) \in C_0(T^*\mathbb{R}^n)$

$$\rho_0(\hat{f})(x, \xi)) = f(x, \xi, 0)$$

We also need to define a product $*_\hbar$ by

$$(\hat{f} *_\hbar \hat{g})(x,y,\hbar) = (2\pi)^{-n} \int \hat{f}(x,z,\hbar)\hat{g}(x+\hbar z, y-z, \hbar)dz \tag{3.3.8}$$

In particular when $\hbar = 0$ we have

$$(\hat{f} *_0 \hat{g})(x,y,0) = (\widehat{fg})(x,y,0)$$

For $\hbar > 0$ we have $\rho_\hbar(\hat{f})\rho_\hbar(\hat{g}) = \rho_\hbar(\hat{f} *_\hbar \hat{g}) \in B(L^2(\mathbb{R}^n))$

Some other useful notes about $\rho_\hbar$ are these

$$Tr(\rho_\hbar(\hat{f})) = \frac{1}{\hbar^n(2\pi)^n} \int \hat{f}(x,0,\hbar)dx = \frac{1}{\hbar^n(2\pi)^n} \int_{T^*\mathbb{R}^n} f(x,\xi,\hbar)dxd\xi \tag{3.3.9}$$

Also

$$\begin{aligned} \delta_{2j-1}(\rho_\hbar(\hat{f})) = \quad & [D_j, \rho_\hbar(\hat{f})] = \quad \rho_\hbar((\widehat{\frac{\partial f}{\partial x_j}})),\ j = 1,\dots,n \\ \delta_{2j}(\rho_\hbar(\hat{f})) = \quad & [M_j, \rho_\hbar(\hat{f})] = \quad \rho_\hbar(\frac{-\hbar}{i}(\widehat{\frac{\partial f}{\partial \xi_j}})),\ j = 1,\dots,n \end{aligned} \tag{3.3.10}$$

So what we have it that

$$\hbar \mapsto \omega(\rho_\hbar(\hat{f}_0), \dots, \rho_\hbar(\hat{f}_{2n}))$$

is continuous for $\hbar > 0$. Which brings us to our fifth point.

We now to show

Lemma 100. *As* $\hbar \to 0$,

$$\omega(\rho_\hbar(\hat{f}_0), \dots, \rho_\hbar(\hat{f}_{2n})) \longrightarrow \varepsilon_0(\rho_0(\hat{f}_0), \dots, \rho_0(\hat{f}_{2n})).$$

Proof. By definition we have

$$\begin{aligned} &\omega(\rho_\hbar(\hat{f}_0), \dots, \rho_\hbar(\hat{f}_{2n})) \\ &= \tfrac{(-1)^n}{n!} \textstyle\sum_{\sigma \in S_n} sgn(\sigma) Tr(\rho_\hbar(\hat{f}_0)\delta_{\sigma(1)}(\rho_\hbar(\hat{f}_1)) \cdots \delta_{\sigma(2n)}(\rho_\hbar(\hat{f}_{2n})) \end{aligned}$$

By the recent properties we've listed we can analyze the trace term and find that we get some nice results. Let's look at the very first term in the sum.

$$\begin{aligned} & Tr(\rho_\hbar(\hat{f}_0)\delta_1(\rho_\hbar(\hat{f}_1)) \cdots \delta_{2n}(\rho_\hbar(\hat{f}_{2n})) \\ = \frac{(-\hbar)^n}{i^n} & Tr(\rho_\hbar(\hat{f}_0 *_\hbar ((\widehat{\partial f_1/\partial x_1}) *_\hbar (\widehat{\partial f_2/\partial \xi_1}) *_\hbar \cdots *_\hbar (\widehat{\partial f_{2n-1}/\partial x_n}) *_\hbar (\widehat{\partial f_{2n}/\partial \xi_n})) \\ & = \frac{(-1)^n}{(2\pi i)^n} \int_{\mathbb{R}^n} (\hat{f}_0 *_\hbar (\widehat{\partial f_1/\partial x_1}) *_\hbar \cdots *_\hbar (\widehat{\partial f_{2n}/\partial \xi_n}))(x,0,\hbar)dx \\ & \to \frac{(-1)^n}{(2\pi i)^n} \int_{T^*\mathbb{R}^n} f_0(\partial f_1/\partial x_1) \cdots (\partial f_{2n}/\partial \xi_n)dxd\xi \end{aligned}$$

We can compute all the terms due to the permutations the same way and then summing them we find

$$\begin{aligned} \lim_{\hbar \to 0} \omega(\rho_\hbar(\hat{f}_0), \dots, \rho_\hbar(\hat{f}_{2n})) &= \frac{1}{(2\pi i)^n n!} \int_{T^*\mathbb{R}^n \times \{0\}} f_0 df_1 \wedge \cdots \wedge df_{2n} \\ &= \varepsilon_0(\rho_0(\hat{f}_0), \dots, \rho_0(\hat{f}_{2n})) \end{aligned}$$

Which is indeed what we wanted to show. □

Point six is nothing more than the definition of the cup product in cyclic homology coupled with the last lemma. For clarity let me state the point six once again.

Lemma 101. *For* $f_0, \dots, f_{2n} \in M_d(\mathcal{S}(T^*\mathbb{R}^n \times \mathbb{R}))$ *the following holds.*

$$\lim_{\hbar \to 0} (\omega \sharp tr)(\rho_\hbar(\hat{f}_0), \dots, \rho_\hbar(\hat{f}_{2n})) = \frac{1}{(2\pi i)^n n!} \int_{T^*\mathbb{R}^n \times \{0\}} tr(f_0 df_1 \wedge \dots \wedge df_{2n})$$

Our seventh point is also quite short. We know that if $P(t)$ is continuous for all t in some interval then $index(P(t)) = index(P(t_0))$ for any chosen t_0. The only possible obstruction we can have to $index(P) = \langle \omega_\hbar, e_\hbar \rangle = \langle \omega_\hbar, e_\hbar \rangle|_{\hbar=0}$ is having a situation where $\omega_\hbar$ or $e_\hbar$ is not continuous at zero. However, to our good fortune, much effort has been expended to construct such $\omega_\hbar$ and $e_\hbar$ so that they are continuous.

Our eighth point is the crux of the proof. This requires some work to set ourselves up for making the claim that this string of equalities is in fact true. Recall that we have $e = (e_\hbar)$ is a continuous field of graph projections. That is; given an elliptic symbol a then $e_a = \begin{pmatrix} (1+a^*a)^{-1} & (1+a^*a)^{-1}a^* \\ a(1+a^*a)^{-1} & a(1+a^*a)^{-1}a^* \end{pmatrix}$ projects $L^2(\mathbb{R}^n; V) \oplus L^2(\mathbb{R}^n; V)$ onto $graph(a)$. This continuity is treated in detail in [ENN]. Now we know

$$index(P) = \langle [e_1] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}], [(2\pi i)^n n! \omega_\hbar] \rangle.$$

In order to get the first equality we need to find some $[e_1^\infty]$ cohomologous to $[e_1]$. This is done by the Murray-von Neumann equivalence. The appeal made by [ENN] is that since $\mathcal{S}(H_{2n+1})$ is stable under the holomorphic functional calculus in $C^*(H_{2n+1})$ there is a continuous field $e^\infty = (e^\infty_\hbar) \in M_2(M_d(S^m)^\sim)$ such that $\|e - e^\infty\| < \varepsilon < 1$. In particular $e \sim e^\infty$.

Remark 102. The discussion of the Heisenberg groups and their C^* algebras has been almost fully omitted from these notes since there is only a small mention in [ENN]. However, it is important to note that [ENN] have a paper which is a predecessor to the one we're currently observing. The topic of that paper was Heisenberg groups and K-theory. In it they constructed continuous fields of C^* algebras and also gave a new proof of Bott periodicity using them. The main point here is that we're guaranteed to have graph projections which are easier to work with.

We've now established

$$\langle [e_1] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}], [(2\pi i)^n n! \omega_\hbar] \rangle = \langle [e_1^\infty] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}], [(2\pi i)^n n! \omega_\hbar] \rangle.$$

The next step is significantly easier. We simply appeal to the fact that $\hbar \mapsto e^\infty_\hbar$ is norm continuous for $\hbar > 0$. And furthermore $\mathcal{K}^\infty$ is stable under holomorphic functional calculus. Hence

$$\langle [e_1^\infty] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}], [(2\pi i)^n n! \omega_\hbar] \rangle = \langle [e_\hbar^\infty] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}], [(2\pi i)^n n! \omega_\hbar] \rangle$$

Again the next step is nearly free once we've established that $e^\infty_\hbar$ is also continuous at zero which is essentially the content of points 5 and 6. Finally we notice that

$e_0^\infty = e_a$ and then we've established

$$\begin{aligned}
&= \langle [e_\hbar^\infty] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}], [(2\pi i)^n n! \omega_\hbar] \rangle \\
&= \langle [e_0^\infty] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}], [(2\pi i)^n n! \varepsilon_0] \rangle \\
&= \langle [e_a] - [\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}], [(2\pi i)^n n! \varepsilon_0] \rangle \\
&= \frac{1}{(2\pi i)^n n!} \int tr(\hat{e}_a (d\hat{e}_a)^{2n})
\end{aligned}$$

The only thing left to do is to show why

$$[e_a] - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = [Ker(P)] - [Ker(P^*)]$$

Theorem 103. *In* $K_0(\mathcal{K}(L^2(\mathbb{R}^n)))$

$$[e_a] - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = [Ker(P)] - [Ker(P^*)]$$

This proof is pulled directly from [ENN]

Proof. The curve

$$[1, \infty) \ni \lambda \mapsto \begin{pmatrix} (1 + \lambda^2 T^* T)^{-1} & (1 + \lambda^2 T^* T)^{-1} (\lambda T^*) \\ \lambda T (1 + \lambda^2 T^* T)^{-1} & \lambda T (1 + \lambda^2 T^* T)^{-1} \lambda T^* \end{pmatrix}$$

Is norm continuous in $M_2(\mathcal{K}(L^2(\mathbb{R}^n; V))^\sim)$. Since the eigenvalue 0 is isolated in the spectrum, by the spectral theorem we have, as $\lambda \longrightarrow \infty$, in the norm topology,

$$(1 + \lambda^2 T^* T)^{-1} \longrightarrow Ker(T^* T),$$

$$(1 + \lambda^2 T^* T)^{-1} (\lambda T^*) \longrightarrow 0,$$

and

$$T(1 + \lambda^2 T^* T)^{-1} \lambda T^* = 1 - (1 + \lambda^2 T^* T)^{-1} \to 1 - Ker(TT^*)$$

Since $Ker(T^*T) = KerT = KerP$ and $Ker(TT^*) = KerT = KerP$ the conclusion follows. □

Remark 104. $T = \overline{P|C_c^\infty(\mathbb{R}^n; V)}$ (the closure of the restriction of P.) And then T^*T is densely defined and self-adjoint and its domain contains $C_c^\infty(\mathbb{R}^n; V)$.
One other consequence of mention is

$$index(P) = Tr((1 + T^*T)^{-1}) - Tr((1 + TT^*)^{-1})$$

3.4. **The Index Theorem of the NC Torus.** Earlier we defined the noncommutative torus as $A_\theta = \mathbb{C}[U, U^*, V, V^*]/(UV = e^{2\pi i\theta} VU)$. More appropriately we can define it is the C^* algebra generated by two unitaries U and V such that $UV = qVU$ where $q = e^{2\pi i\theta}$. We're now interested in index theory related to A_θ. This means we should consider a smooth structure on A_θ and moreover we'll want to consider a notion of elliptic operators on the smooth space.

Definition 105. The smooth structure on A_θ is a subalgebra denoted $\mathcal{A}_\theta$ is given by

$$\mathcal{A}_\theta = \{\sum_{n,m} a_{n,m} U^n V^m : a \in \mathcal{S}(\mathbb{Z}^2)\}$$

where $a \in \mathcal{S}(\mathbb{Z}^2)$ means

$$(|n|^k + |m|^k) \cdot |a_{n,m}| < \infty \quad \forall k > 0$$

This subalgebra is stable under the holomorphic functional calculus. Hence $\mathcal{A}_\theta \subsetneq A_\theta$ is a pre-C^*-algebra. In order to consider elliptic theory on $\mathcal{A}_\theta$ we need to consider a space where it can act. We have the following classification theorem.

Theorem 106. *Let E be a finitely generated projective module over $\mathcal{A}_\theta$; then E is free ($\mathcal{A}_\theta^n$ some $n > 0$) or $E \cong \mathcal{S}(\mathbb{R}) \otimes K$. Where K is a finite dimensional representation of the Heisenberg commutation relations for $\mathbb{Z}/(m)$ some $m \in \mathbb{Z}$.*

The rest of the exposition will be centered on $E = \mathcal{S}(\mathbb{R})$. The (right) module structure of E is given by

$$\begin{aligned}(\xi \cdot U)(s) &= \xi(s+\theta) \quad \forall s \in \mathbb{R}, \xi \in \mathcal{S}(\mathbb{R}) \\ (\xi \cdot V)(s) &= e^{2\pi i s}\xi(s) \quad \forall s \in \mathbb{R}, \xi \in \mathcal{S}(\mathbb{R})\end{aligned} \tag{3.4.1}$$

Now what we'll do is use a dictionary between geometry and algebra. In particular we will use the correspondence

$$\textit{Bundle on } M \overset{\simeq}{\longleftrightarrow} \textit{f.g.p. module on } C^\infty(M).$$

With this in mind we can define connections on $\mathcal{S}(\mathbb{R})$. In fact we have derivations on $\mathcal{A}_\theta$ given by

$$\begin{cases} \delta_1(U) = 2\pi i U \\ \delta_1(V) = 0 \\ \delta_2(U) = 0 \\ \delta_2(V) = 2\pi i V \end{cases}$$

Which lift to sections $\xi \in E$ as

$$\begin{cases} (\nabla_1 \xi)(s) = -2\pi i \frac{s}{\theta}\xi(s) \\ \\ (\nabla_2 \xi)(s) = \frac{d\xi}{ds} \end{cases}$$

Connes points out that these forms are not relevant. Once we've given their explicit formulations we forget about them completely in favor of operators

$$D : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}), (D : E \to E)$$

of the form

$$D = \sum_{\alpha,\beta} C_{\alpha,\beta} \nabla_1^\alpha \nabla_2^\beta$$

Here our coefficients $C_{\alpha,\beta}$ are operators of order 0 on E. That is to say

$$C_{\alpha,\beta} \in End_{\mathcal{A}_\theta}(E)$$

Remark 107. The space $End_{\mathcal{A}_\theta}(E)$ is isomorphic to the space $\mathcal{A}_{\theta'}$ where $\theta' = 1/\theta$. We can view this space as the C^* algebra generated by unitaries X, Y s.t.

$$XY = e^{2\pi i \theta'} YX$$

and more explicitly

$$(X\xi)(s) = \xi(s+1)$$
$$(Y\xi)(s) = e^{-2\pi is/\theta}\xi(s)$$

Which leads us to the fact that D can be written as an element of the algebra of operators on $\mathcal{S}(\mathbb{R})$ generated by

1. Multiplication $\hat{s}$: $\hat{s}\xi(s) = s\xi(s)$
2. Differentiation : d/ds
3. Multiplication by $e^{2\pi is/\theta}$
4. Finite difference Δ : $\Delta\xi(s) = \xi(s+1) - \xi(s)$

Since $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ is dense this algebra extends to an algebra of operators on $L^2(\mathbb{R})$.

It is always good to keep around the ideas from classical geometry for intuition. So we note that in the commutative case (i.e. $\theta = 1$) The operators D are exactly the differential operators on the nontrivial line bundle on T^2. For the next few definitions and theorems let's keep the ideas of classical geometry and topology around for intuition and analogy.

Definition 108. a For $n \in \mathbb{Z}$ and $n \geq 0$ we say that D is of order $\leq n$ if D can be written as

$$D = \sum_{\alpha+\beta\leq n} C_{\alpha,\beta}\nabla_1^\alpha\nabla_2^\beta; \quad C_{\alpha,\beta} \in End_{\mathcal{A}_\theta}(E)$$

b For an operator of order n we have the symbol map

$$\sigma_n(D) : S^1 \to \mathcal{A}_{\theta'}$$

given by

$$\sigma_n(D)(\eta_1, \eta_2) = \sum_{\alpha+\beta=n} C_{\alpha,\beta}\eta_1^2\eta_2^2$$

where $\eta_1^2 + \eta_2^2 = 1$.

c D is *elliptic* if $\sigma_n(D)$ is invertible in $\mathcal{A}_{\theta'}$

c-bis D is *elliptic* if $\sigma_n(D)$ is invertible in $L^2(\mathbb{R})$.

Notice that (c) and (c-bis) are equivalent since $\mathcal{A}_{\theta'}$ is stable under the holomorphic functional calculus and its norm closure is $L^2(\mathbb{R})$.

We are now ready to begin the actual index theory on the noncommutative torus.

Theorem 109. *Let* $D = D = \sum C_{\alpha,\beta}\nabla_1^\alpha\nabla_2^\beta$ *be elliptic then*

(1) $ker(D)$ *is finite dimensional*

(2) If $\xi \in L^2(\mathbb{R}) \cap ker(D)$ *then* $\xi \in \mathcal{S}(\mathbb{R})$

This is a direct analogue of a pair of theorems dealing with elliptic regularity. Recall

Theorem 110. *If $A \in \Psi^m(M)$ is elliptic and M is compact then*

(1) A is Fredholm.

(2) $ker(A) = \{u \in L^2(M) : Au = 0\} = \{u \in C^\infty(M) : Au = 0\}$

N.B. This theorem can be found in chapter 5 of Melrose.

While we're still thinking in the classical setting let's also recall that the index of an operator is given by the pairing of a cycle with a cocycle.

Remark 111. An interesting historical note is that in developing his version of cyclic homology Connes was looking for a receptacle to the classical Chern character in the noncommutative case.

Now we expect our index formula for D to involve a cocycle on $C^\infty(S^1, \mathcal{A}_{\theta'})$ since this is where we're defining our principal symbol. From here we only need to define our cocycles and state the main theorem.

Our first cocycle is the canonical trace on $\mathcal{A}_{\theta'}$

$$\tau_0(\sum C_{\alpha,\beta} X^\alpha Y^\beta) = a_{0,0}.$$

Our second cocycle is given by

$$\tau_2(a^0, a^1, a^2) = \tau_0(a^0(\delta'_1(a^1)\delta'_2(a^2) - \delta'_1(a^2)\delta'_2(a^1))) = \tau_0(a^0(\delta'_1 \wedge \delta'_2)(a^1, a^2)).$$

Where δ'_i are derivations on $\mathcal{A}_{\theta'}$ given by

$$\begin{cases} \delta'_1(X) = 2\pi i X \\ \delta'_1(Y) = 0 \\ \delta'_2(X) = 0 \\ \delta'_2(Y) = 2\pi i Y \end{cases}$$

Notice now that these cocycles are only defined on $\mathcal{A}_{\theta'}$. We need cocycles on $C^\infty(S^1, \mathcal{A}_{\theta'})$. However we have a convenient equivalence

$$C^\infty(S^1, \mathcal{A}_{\theta'}) = C^\infty(S^1) \hat{\otimes} \mathcal{A}_{\theta'}$$

which means we only need to know the cocycles on $C^\infty(S^1)$. Indeed we know these since we can go back to our equivalence given by HKR. Let $\rho = [S^1]$ be the fundamental class of S^1 (i.e. the 1-cocycle given by

$$\rho(f^0, f^1) = \int_{S^1} f^0 df^1.$$

The cocycles we want are now defined as

$$\tau_1 = \rho \sharp \tau_0 \in Z^1(C^\infty(S^1, \mathcal{A}_{\theta'})) \tag{3.4.2}$$

$$\tau_3 = \rho \sharp \tau_2 \in Z^3(C^\infty(S^1, \mathcal{A}_{\theta'})) \tag{3.4.3}$$

The cup product on cyclic cohomology yields the following formulas

$$\tau_1(\sigma_0, \sigma_1) = \int_{S^1} \tau_0(\sigma_0(t) \frac{d}{dt} \sigma_1(t)) dt \tag{3.4.4}$$

$$\tau_3(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = \int_{S^1} \tau_0(\sigma_0 d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3) dt \tag{3.4.5}$$

Of course $d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3$ merits some explanation. It is the element of $C^\infty(S^1, \mathcal{A}_{\theta'})$ given by

$$t \mapsto \sum_{\pi \in S_3} sgn(\pi) \partial_{\pi(1)} \sigma_1(t) \partial_{\pi(2)} \sigma_2(t) \partial_{\pi(3)} \sigma_3(t)$$

and

$$(\partial_i\sigma)(t) = \left\{ \begin{array}{ll} \delta_1'(\sigma(t)) & i=1 \\ \delta_2'(\sigma(t)) & i=2 \\ \frac{d}{dt}(\sigma(t)) & i=3 \end{array} \right\} for\ t \in S^1,\ \sigma \in C^\infty(S^1, \mathcal{A}_{\theta'}).$$

We are now able to state the analogue of Atiyah-Singer.

Theorem 112. *Let D be elliptic and $\sigma(t) = \sum_{\alpha+\beta=n} C_{\alpha,\beta}(cos(t))^\alpha(sin(t))^\beta$ be its principal symbol. Then*

$$index(D) = \frac{1}{\theta}\frac{1}{(2\pi i)^2}\frac{1}{6}\tau_3(\sigma^{-1},\sigma,\sigma^{-1},\sigma) - \frac{1}{2\pi i}\tau_1(\sigma^{-1},\sigma) \tag{3.4.6}$$

The rest of this section will be some exposition on the formulas and a few corollaries.

Let's mimic a Laurent series for $a \in \mathcal{A}_{\theta'}$

$$a = \sum_{n\in\mathbb{Z}} f_n X^n.$$

Here, of course $f_n \in C^\infty(\mathbb{R}/\theta\mathbb{Z})$ and certainly knowing the structure of $\mathcal{A}_{\theta'}$ we have

$$f_n = \sum_m a_{n,m} Y^m.$$

Given another element $b = \sum g_n X^m$ we have the rule

$$a \cdot b = \sum f_n \gamma^n(g_n) X^{n+m}$$

essentially saying that in this guise the multiplication acts as a crossed product. Here γ acts by

$$\gamma(g)(s) = g(s+1)\ \ \forall s \in \mathbb{R},\ g \in C^\infty(\mathbb{R}/\theta\mathbb{Z}).$$

In this notation our trace τ_0 becomes

$$\tau_0(\sum f_n X^n) = \frac{1}{\theta}\int_0^\theta f_0(s)ds$$

In a similar way we can write $\sigma \in C^\infty(S^1, \mathcal{A}_{\theta'})$ as a Laurent series

$$\sigma = \sum_{n\in\mathbb{Z}} \sigma_n X^n.$$

Now each σ_n is not simply a function on $\mathbb{R}/\theta\mathbb{Z}$ instead it is a doubly periodic function

$$\sigma_n(t,s); t \in S^1,\ s \in \mathbb{R}/\theta\mathbb{Z}$$

Now in the current guise we have the derivations $\partial_1, \partial_2, \partial_3$ given by

$$\partial_1\sigma = \sum_{n\in\mathbb{Z}} 2\pi i n \sigma_n X^n$$

$$(\partial_2\sigma)_n(t,s) = -\theta\frac{\partial}{\partial s}\sigma_n(t,s)$$

$$(\partial_3\sigma)_n(t,s) = \frac{\partial}{\partial t}\sigma_n(t,s)$$

By which we find the formulas for τ_1 and τ_3 to be

$$\tau_1(\sigma^0, \sigma^1) = \frac{1}{\theta} \int_0^\theta \int_{S^1} (\sigma^0 \partial_3 \sigma^1)_0(t, s) dt ds \tag{3.4.7}$$

$$\tau_3(\sigma^0, \sigma^1, \sigma^2, \sigma^3) = \frac{1}{\theta} \int_0^\theta \int_{S^1} (\sigma^0 d\sigma^1 \wedge d\sigma^2 \wedge d\sigma^3)_0 dt ds \tag{3.4.8}$$

The main theorem may be interpreted in this way if one so chooses.

4. K-THEORY

Here I will only present the necessary material for understanding the previous sections.

4.1. **Algebraic K_0 and K_1.**

The Grothendieck group K_0 has been well studied. Here I just intend to make a presentation sufficient to understand what it is and one that leads naturally to the topological definition.

Definition 113. Let A be a ring (not necessarily commutative) and let P be a finitely generated projective (f.g.p) module over A. Denote by $[P]$ the isomorphism class of the module P. That is to say if $P \cong P'$ then $[P] = [P']$. We define the group $K_0(A)$ by

$$K_0(A) = \frac{\mathcal{F}(f.g.p.\ over\ A)}{[P] + [P'] - [P \oplus P']} \tag{4.1.1}$$

This definition tells us that we allow *formal* differences of f.g.p. modules. In fact every element of $K_0(A)$ can be written $[P] - [Q]$ for some f.g.p. modules P and Q. Moreover we may take Q to be free. In the case that A is commutative the tensor product provides a ring structure on $K_0(A)$ by

$$[P] \times [Q] = [P \otimes_A Q].$$

Notice that in the case A is not commutative we are not guaranteed a ring because the tensor product does not guarantee a f.g.p. module over A it merely guarantees an abelian group. However K_0 is still a group. In fact an abelian group.

We can see further that K_0 is a functor from the category of rings to abelian groups by the following diagram

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f"'] & K_0(A_1) \arrow[d, "f_*"] \\
A_2 \arrow[r] & K_0(A_2)
\end{tikzcd}$$

The easiest examples of $K_0(A)$ involve algebras A that only admit free projective modules (i.e. every projective module is isomorphic to A^n for some n.) In this case $K_0(A) \cong \mathbb{Z}$ and the isomorphism is induces by the rank map. Examples of such algebras are fields, local rings (think Nakayama lemma), and $\mathbb{Z}$ by virtue of the classification of modules over a P.I.D. The lead in to topological K_0 sometimes denoted K^0 is that $C(X)$ is a commutative ring for paracompact spaces. We'll discus this later.

The algebraic group $K_1(A)$ has a simple definition

$$K_1(A) := GL(A)/[GL(A), GL(A)] \tag{4.1.2}$$

where

$$GL(A) := \varinjlim GL_n(A).$$

One simple result we can obtain is that

$$K_1(A) = H_1(GL(A), \mathbb{Z})$$

where $H_1(G, \mathbb{Z})$ is group homology. The topological interpretation of this statement is

$$K_1(A) = H_1(BGL(A), \mathbb{Z})$$

where $BGL(A)$ is the classifying space of $GL(A)$

4.2. **Topological K-theory.** A proper systematic study of topological K-theory should probably begin with vector bundles and their properties. Here I will just mention a few and draw the diagram for pullbacks.

Given a topological space X and two vector bundles E, F over X we have several operations.

- tensor product $E \otimes F \cong F \otimes E$
- direct sum $E \oplus F \cong F \oplus E$
- distributivity of tensor over direct sum $E \otimes (F_1 \otimes F_2) = (E \otimes F_1) \oplus (E \otimes F_2)$
- dual E^*
- $Hom(E, F) = E^* \otimes F$
- exterior powers $\Lambda^k(E \oplus F) = \bigoplus_{i+j=k} \Lambda^i(E) \otimes \Lambda^j(F)$
- pullbacks: Let $f : Y \to X$ be a continuous map. Then f^*E is defined by

$$f^*E = \{(e, y) : e \in E, y \in Y, \pi(e) = f(y)\}$$

We can see the pullback by the following diagram.

$$\begin{tikzcd}
(e,y) \arrow[rrr, mapsto] \arrow[ddd] & & & e \arrow[ddd] \\
& f^*E \arrow[r] \arrow[d, "\pi"] & E \arrow[d, "\pi"'] & \\
& Y \arrow[r, "f"] & X & \\
y \arrow[rrr, mapsto] & & & x
\end{tikzcd}$$

The nice property regarding pullbacks is that they are natural transformations. More precisely, if $T : E \to F$ is a vector bundle map then $f^*T(E) = T(f^*E)$.

We are now in a position to define the group $K(X)$ for a paracompact topological space X. In most cases we'll consider X to be a manifold and in particular a compact manifold.

Definition 114. The group $K(X)$ is defined by

$$K(X) := K_0(C(X))$$

where $K_0(C(X))$ is the algebraic K_0.

Perhaps it is the content of a lemma or even a theorem, but the reason we make all the fuss about vector bundles before defining $K(X)$ and then use an algebraic definition is that there is an equivalence of categories or a dictionary between geometry and algebra that says (at the level of vector bundles)

Vector Bundles on $X \overset{\cong}{\leftrightarrows}$ finitely generated projective modules on $C(X)$

Moreover when X is a smooth manifold $C^\infty(X)$ is dense in $C(X)$ so we can refine our definition even further

$$K(X) = K_0(C^\infty(X))$$

In the section on K_0 we remarked that any element can be written in the form $[P]-[Q]$ and we may take Q to be free. This essentially follows from the fact that projective modules are summands of free modules. When we take this to vector bundles we get the same thing. But quickly recall that by a trivial vector bundle of rank n over X we mean $X \times \mathbb{R}^n$ or in the case of complex K theory as we'll be discussing (for Bott periodicity) $X \times \mathbb{C}^n$.

Proposition 115. *Every element of* $K(X)$ *can be written in the form* $[E]-[\underline{n}]$ *where* $\underline{n}$ *is a trivial bundle of rank* n*.*

Proof. Let E, F, G be vector bundles over X and $F \oplus G = \underline{n}$ for some n. Then

$$[E]-[F]=[E]+[G]-([F]+[G])=[E \oplus G]-[F \oplus G]=[H]-[\underline{n}]$$

$\square$

Definition 116. We call two bundles E, F *stably equivalent* is there is a suitable bundle $\underline{n}$ s.t. $E \oplus \underline{n} \cong F \oplus \underline{n}$

Let us restrict our attention at this point to compact manifolds. Let $Vect_n(X)$ denote the bundles of rank n on X up to isomorphism.

Lemma 117. $K(X) \cong \varinjlim \mathbb{Z} \times Vect_n(X)$

From here we'll make a few more definitions and state come consequences. The goal is two-fold.

1. Give a homotopy theoretic definition of $K(X)$
2. State the Bott periodicity theorem.

For the first goal let us recall

Definition 118. The *complex Grassmanian* $G_{\mathbb{C}}(n, N)$ is defined to be all n-dimensional complex subspaces passing through the origin in $\mathbb{C}^{n+N}$. Some may also use the notation to mean copies of $\mathbb{C}^n \subset \mathbb{C}^N$.

Notice that in the first notation $G_{\mathbb{C}}(1, N) = \mathbb{C}P^N = \{$complex lines passing through the origin in $\mathbb{C}^{N+1}\}$.

In chapter 1 of Atiyah (and other places of course) we have a proposition that says for a (manifold) X,

$$\varinjlim_{m}[X, G_{\mathbb{C}}(n, m)] = Vect_n(X). \tag{4.2.1}$$

Where $[X, M]$ denotes homotopy classes of maps from X to M. A consequence of algebraic topology is that we also have an equivalence

$$BU(n) = \varinjlim_{m} G_{\mathbb{C}}(n, m)$$

where $BU(n)$ is the classifying space for the unitary group $U(n)$.

Let us also define

$$U := \varinjlim_{n} U(n)$$

Then our proposition may be restated

$$\begin{aligned} [X, BU] &\cong \varinjlim_n [X, BU(n)] \\ &\cong \varinjlim_n \varinjlim_m [X, G_{\mathbb{C}}(n,m)] \\ &\cong \varinjlim_n Vect_n(X) \\ &\cong \tilde{K}(X) \end{aligned} \tag{4.2.2}$$

Let us also recall that $\tilde{K}(X)$ is the reduced K theory in which we only consider nontrivial bundles, or equivalently we say $[E] = [F] \in \tilde{K}(X)$ if there exist $n, m \in \mathbb{Z}_+$ such that

$$[E] + [\underline{n}] = [F] + [\underline{m}].$$

Hence in this case all trivial bundles are considered equivalent regardless of rank.

This leads us to several propositions rolled into one

Proposition 119. *(1)* $K(X) = [X, \mathbb{Z} \times BU]$

(2) $\tilde{K}(SX) = [X, U]$ *Where* SX *is the suspension of* X

(3) $U \sim \Omega BU$ *where* ΩX *denotes the loop space on* X.

(4) $K^{-n}(X) \cong [X, \Omega^n BU] = [X, \Omega^{n-1} U]$ *This may essentially be taken as the definition of* $K^{-n}(X)$.

Remark 120. A quick note on suspensions is that we're usually considering $X \in Ob(Top_*)$ (i.e. we consider X to have distinguished base point. In this case our suspension is actually the reduced suspension.

We also write $S^1 \wedge X = SX$ for ease of notation. Truly speaking the suspension of X is indeed the smash product with a circle. Furthermore the spheres smash very nicely

$$S^n \wedge S^m \simeq S^{n+m}$$

so we have

$$S^n \simeq \underbrace{S^1 \wedge \cdots \wedge S^1}_{n-copies}$$

Thus we define the n suspension of X as $S^n \wedge X$ or simply $S^n X$. Then the actual definition of $K^{-n}(X)$ is

$$\tilde{K}^{-n}(X) = \tilde{K}(S^n X) \quad X \in Ob(Top_*)$$

Let $Y \subseteq X$ then we define the relative K theory of (X, Y) as

$$K(X, Y) = \tilde{K}(X/Y) \tag{4.2.3}$$

Now we have the following definitions

Definition 121. For $n \geq 0$

$$\begin{array}{lllllll} \tilde{K}^{-n}(X) & = & \tilde{K}(S^n X) & & & for & X \in Ob(Top_*) \\ K^{-n}(X, Y) & = & \tilde{K}^{-n}(X/Y) & = & \tilde{K}(S^n(X/Y)) & for & (X, Y) \in Ob((Top)^2) \\ K^{-n}(X) & = & K^{-n}(X, \emptyset) & = & \tilde{K}(S^n(X^+)) & for & X \in Ob(Top) \end{array}$$

And X^+ is the compactification of X.

Let us mention a few propositions about products and then we'll be ready to state the big theorem.

Proposition 122. *If Y is a retract of X then for all $n \geq 0$ the short exact sequence*

$$K^{-n}(X,Y) \to K^{-n}(X) \to K^{-n}(Y)$$

is split and then

$$K^{-n}(X) \cong K^{-n}(X,Y) \oplus K^{-n}(Y) \tag{4.2.4}$$

Proposition 123. *If X, Y are two spaces with basepoints then the projections*

$$\pi_X : X \times Y \to X$$

and

$$\pi_Y : X \times Y \to Y$$

induce isomorphisms for all $n \geq 0$.

$$\tilde{K}^{-n}(X \times Y) \cong \tilde{K}^{-n}(X \wedge Y) \oplus \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y) \tag{4.2.5}$$

We are now ready to state the Bott periodicity theorem.

Theorem 124.

$$K(X) \cong K^{-2}(X) \tag{4.2.6}$$

The actual proof of Bott's theorem is rather lengthy. I shall not present it all here. I will however mention some key ideas and then read off some consequences.

The main idea of the proof is to construct two maps

$$\alpha : K^{-2}(X) \to K(X)$$
$$\beta : K(X) \to K^{-2}(X)$$

then show that they are inverses. Bott had originally shown a homotopy equivalence

$$U \sim \Omega^2 U.$$

However in view of $U \sim \Omega BU$ we can replace the above equivalence with

$$\Omega U \sim \mathbb{Z} \times BU$$

And in light of this we can look for maps

$$\alpha : \Omega U \to \mathbb{Z} \times BU$$
$$\beta : \mathbb{Z} \times BU \to \Omega U.$$

The idea is to show that these α, β are homotopy inverses of each other. β is essentially $\lim_{n\to\infty} \beta_n$ where

$$\beta_n : G_{\mathbb{C}}(n,n) = \frac{U(2n)}{U(n) \times U(n)} \to \Omega U(2n)$$

is an embedding.

The construction of α requires yet another model for $\mathbb{Z} \times BU$. Cutting right to the chase; there is a homotopy equivalence

$$Fred(\mathcal{H}) \sim \mathbb{Z} \times BU$$

in light of an isomorphism

$$index : [X, Fred(\mathcal{H})] \to K(X).$$

Then it suffices to construct

$$\alpha : \Omega U \to Fred(\mathcal{H}).$$

Now for any $f : S^1 \to U(n)$ we can define T_f a Toeplitz operator acting on $Hol(D^2, \mathbb{C}^n)$ by multiplication by f. Then $\alpha(f) = T_f$.

The construction of β involves looking at $S^2 = \mathbb{C}_\infty = \mathbb{C}P^1$. Let us briefly note that there is a line bundle H over $\mathbb{C}P^1$ called the canonical line bundle (by topologists) that generates the nontrivial copy of $\mathbb{Z}$ in $K(S^2)$. That is

$$b := [H] - [1] \in \tilde{K}(S^2) = K^{-2}(pt).$$

One can find an explicit formula for b for example in Varilly.

$$b = \frac{1}{1 + z\bar{z}} \begin{pmatrix} z\bar{z} & \bar{z} \\ z & 1 \end{pmatrix}$$

We have that the bundle in class $m \in \tilde{K}(S^2)$ is given by

$$b_m = \frac{1}{1 + (z\bar{z})^m} \begin{pmatrix} (z\bar{z})^m & \bar{z}^m \\ z^m & 1 \end{pmatrix}$$

Remark 125. There is a lot to actually prove, but Atiyah has a nice exposition of it, so I leave the reader to Atiyah's book on K-theory for the full proof.

Theorem 126. *The tensor product of bundles induces an isomorphism*

$$K(S^2) \otimes K(X) \xrightarrow{\sim} K(S^2 \times X) \tag{4.2.7}$$

We now see that we have $K-n(X) \cong K^{-n-2}(X)$ by virtue of periodicity. With this in mind we can define $K^n(X)$ for all integers n. And in fact we make the following definition.

Definition 127. The graded abelian group $K^*(X)$ is defined to be

$$K^*(X) = K^0(X) \oplus K^1(X).$$

Bearing all this in mind we arrive at a somewhat miraculous *six term* exact sequence

$$\begin{tikzcd}
K^0(X,Y) \arrow[r] & K^0(X) \arrow[r] & K^0(Y) \arrow[d] \\
K^1(Y) \arrow[u] & K^1(X) \arrow[l] & K^1(X,Y) \arrow[l]
\end{tikzcd}$$

4.3. **The Chern Character.** The traditional Chern character is a characteristic class for a vector bundle on a manifold. It is however easy to formulate a Chern character for algebras. The assumptions we'll have for now are that A is a commutative algebra with module M which in this case will be considered as a right module.

Definition 128. A connection on M is a $\mathbb{C}$-linear map usually denoted ∇ and

$$\nabla : M \otimes_A \Omega_A^n \to M \otimes_A \Omega_A^{n+1}$$

such that

$$\nabla(x\omega) = \nabla(x)\omega + (-1)^n x d\omega$$

where $x \in M \otimes \Omega^n$ and $\omega \in \Omega^p$

It is clear then that ∇ is completely determined by its action on $M \otimes \Omega^0 = M$. We should also notice that

$$\nabla^2 : M \otimes \Omega^n \to M \otimes \Omega^{n+2}$$

is Ω^* linear.

$$\begin{array}{rc} & \nabla^2(x\omega) = \nabla \circ \nabla(x\omega) \\ = & \nabla(\nabla(x)\omega + (-1)^n x d\omega) \\ = & \nabla^2(x)\omega + (-1)^{n+1}\nabla(x)d\omega + (-1)^n\nabla(x)d\omega + xd^2\omega \\ = & \nabla^2(x)\omega \end{array}$$

Now it makes sense to define ∇^{2n} for any $n \geq 0$ and that leads us to the definition

Definition 129. The *Chern character* of a module M is

$$ch(M, \nabla) = tr(exp(\nabla^2)).$$

Where

$$exp(\nabla^2) = I + \sum_{n\geq 1} \nabla^{2n}/(n!)$$

Remark 130. In the complex geometry case it is customary to normalize

$$ch(M, \nabla) = tr(exp(\nabla^2/2\pi i))$$

Furthermore it is well known that $ch(M, \nabla)$ is independent of the connection chosen. For this reason, it is customary to use the Levi-Civita connection.

Definition 131. Let M be a finitely generated projective (f.g.p.) A-module and d the exterior differential on $\Omega^*(A)$. That is $M \cong eA^r$ where e is an idempotent (we know this from $K_0(A)$). Then we define the *Levi-Civita* connection on M as the composition of the following diagram

$$M \otimes_A \Omega_A^* \hookrightarrow A^r \otimes \Omega_A^* \overset{(d,\ldots,d)}{\longrightarrow} A^r \otimes \Omega_A^{*+1} \overset{e\otimes id}{\longrightarrow} M \otimes \Omega_A^{*+1}$$

It is important to note that this may not be the standard name for this connection, however it is analogous to the Levi-Civita connection in differential geometry where $M = C^\infty(X)$ for a smooth manifold X.

Given this connection on M we can compute with zeal and find that

$$ch(Im(e), \nabla_e) = [\frac{1}{n!} tr(e(de)^{2n})].$$

Theorem 132. *For any commutative and unital algebra A*

$$ch_0 : K_0(A) \longrightarrow H_{DR}^{even}(A)$$

defines a ring homomorphism.

Proof. We only need to check three properties

(1) $M \cong M' \implies ch(M) = ch(M')$

(2) $ch(M_1 \oplus M_2) = ch(M_1) + ch(M_2)$

(3) $ch(M_1 \otimes M_2) = ch(M_1)ch(M_2)$

Each property is quickly verified by noting

(1) We only need to consider the following commutative diagram

$$\begin{tikzcd}
End_A(M) \otimes \Omega^*_A \arrow[r] \arrow[d, "tr"'] & End_A(M') \otimes \Omega^*_A \arrow[d, "tr"'] \\
\Omega^*_A \arrow[r, "id"] & \Omega^*_A
\end{tikzcd}$$

(2) The direct sum of connections is the connection of a direct sum

(3) $H^{even}_{DR}(A)$ can be identified with $HC^-_*(A)$ (see Loday §8.2 for more details)

□

Now we have several ways of extending the Chern character as a ring homomorphism. Ostensibly we view ch as a map from K_0 to H^{even}_{DR}, but in fact we can do much more than this. As we know, for a smooth algebra A we have

$$HC_n(A) = \Omega^n_A / d\Omega^{n-1}_A \oplus \bigoplus_k H^{n-2k}_{DR}(A)$$

So it makes sense to restrict ch to map into HC_{2n}. We can achieve this in at least two ways

(1) By mapping into $H^\lambda_{2n}(A)$

(2) By producing a $2n$-cycle in $Tot(CC(\mathcal{M}(A)))$

Let's begin by examining the first case. Let $R := M_r(A)$ be the ring of matrices with coefficients in A. Let $e \in M_r(A)$ be an idempotent. Then for $e^{n+1} \in C_n(R) = R^{\otimes n+1}$ we have

$$b(e^{\otimes n}) = \begin{cases} e^{\otimes n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Equivalently

$$b(e^{\otimes n+1}) = \begin{cases} e^{\otimes n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Now consider working in $C^\lambda_n(R) = R^{\otimes n+1}/(1-t)$ where t is the cyclic operator. We see that $e^{\otimes n} = (-1)^{n-1} e^{\otimes n} \in C^\lambda_{n-1}(R)$ and hence $e^{\otimes 2n-1} = 0$. That is to say $e^{2n+1} \in Z^\lambda_{2n}(R)$. This means that applying the generalized trace map we get an element of $C^\lambda_*(A)$ which will produce our Chern character.

Definition 133. The map $ch^\lambda_{0,n} : K_0(A) \to H^\lambda_{2n}(A)$ is given by

$$ch^\lambda_{0,n}([e]) = tr((-1)^n e^{\otimes 2n+1})$$

This map is well defined and functorial in A.

Now let us move to the second method.

Definition 134. With the same assumptions on A as before we define $ch_{0,n} : K_0(A) \to HC_{2n}(A)$ by

$$ch_{0,n}([e]) = tr(c(e))$$

We define $c(e)$ as follows

Definition 135. For any idempotent $e \in \mathcal{M}(A)$ let

$$\begin{aligned} y_i &:= (-1)^i \frac{(2i)!}{i!} e^{\otimes 2i+1} \in \mathcal{M}(A)^{\otimes 2i+1} \\ z_i &:= (-1)^{i-1} \frac{(2i)!}{2(i!)} e^{\otimes 2i} \in \mathcal{M}(A)^{\otimes 2i} \end{aligned}$$

and let

$$c(e) := (y_n, z_n, y_{n-1}, z_{n-1}, \dots, y_0) \in \mathcal{M}(A)^{\otimes 2n+1} \oplus \mathcal{M}(A)^{\otimes 2n} \oplus \cdots \oplus \mathcal{M}(A)$$

Lemma 136. *By picking n and examining the appropriate $c(e)$ we see*

$$c(e) \in Z_{2n}(Tot(CC(\mathcal{M}(A)))).$$

Furthermore

$$Im(c(e)) = (y_n, y_{n-1}, \dots, y_1) \in Tot(BC(\mathcal{M}(A))$$

Proof. We only need to check that $b(y_i) = -(1-t)(z_i)$ and $b'(z_i) = N(y_{i-1}$. Then this is just an exercise in knowing definitions.

$$t(e^{\otimes 2i}) = -e^{\otimes 2i}$$

$$b(-2e^{\otimes 2i+1}) = -2e^{\otimes 2i} = -(1-t)e^{\otimes 2i}$$

$$b'(-ie^{\otimes 2i}) = -ie^{\otimes 2i-1} = N(e^{\otimes 2i-1})$$

Visually it is helpful to think of $Tot(CC(\mathcal{M}(A)))$. Here is an example for $n = 2$.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\bigcirc \arrow[d, "b"'] & \arrow[l, "1-t"'] \arrow[d, "-b'"'] & \arrow[l, "N"'] \arrow[d, "b"'] & \cdots \\
\arrow[d, "b"'] & \bigcirc \arrow[l, "1-t"'] \arrow[d, "-b'"'] & \arrow[l, "N"'] \arrow[d, "b"'] & \cdots \\
 & \arrow[l, "1-t"'] & \bigcirc \arrow[l, "N"'] & \cdots
\end{tikzcd}$$

□

We also have the following property

$$S \circ ch_{0,n}([e]) = ch_{0,n-1}([e]).$$

Since the periodicity operator S acts by $(y_n, z_n, \dots, y_0) \mapsto (y_{n-1}, z_{n-1}, \dots, y_0)$.

Corollary 137. *Recalling the projection $p : Tot(CC(A)) \to C_*^\lambda(A)$ we get that*

$$p_* \circ ch_{0,n} = (-1)^n \frac{(2n)!}{n!} ch_{0,n}^\lambda$$

Now on to examples.

Example 138. Consider $A = \mathbb{C}[x,y,z]/(x^2+y^2+z^2=1)$, the algebraic 2-sphere. Then $K_0(A) = \mathbb{Z} \oplus K_0(\mathbb{C}) = \mathbb{Z} \oplus \mathbb{Z}$ where the first copy of $\mathbb{Z}$ is generated by $e = (1+p)/2$ and

$$p = \begin{bmatrix} x & y+iz \\ y-iz & -x \end{bmatrix}$$

We have a few ways to compute $ch_{0,2}([e])$. Let us use $ch_{0,2}^\lambda([e]) = tr((-1)^2 e^{\otimes 3})$ with the generalized trace map.
Recall $tr : \mathcal{M}_r(A) \otimes \mathcal{M}_r(A) \to A \otimes A^{\otimes n}$ is given by

$$tr(\alpha \otimes \cdots \otimes \eta) = \sum_{all(i_0,\ldots,i_n)} \alpha_{i_0,i_1} \otimes \beta_{i_1,i_2} \otimes \cdots \otimes \eta_{i_n,i_0}$$

Now $tr(e^{\otimes 3})$ becomes a <u>64 term</u> beast when we expand it all out. However there are many cancelations and once we apply the HKR map to the remaining terms we arrive at

$$\chi(tr(e^{\otimes 3})) = -\frac{i}{4}(xdydz + ydzdx + zdxdy)$$

and then pushing over with p_* we get

$$-\frac{i}{2}(xdydz + ydzdx + zdxdy) \in \Omega_A^2/d\Omega_A^1 \subset HC_2(A).$$

It should also be noted that this is $tr(\frac{1}{8}pdpdp)$ which is the volume form of the 2-sphere in the complex case.

Remark 139. Here is an explicit formula for $ch(P)$
(4.3.1)

$$ch(P) = P + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} (P - 1/2) \otimes \underbrace{P \otimes \cdots \otimes P}_{2k-times} \in \prod_{k=0}^{\infty} C_k(A,A) = \prod_{k=0}^{\infty} A \otimes A^{\otimes k}$$

In particular if we're dealing with the algebraic 2-sphere and e as above then as $e = 1/2 + p/2$ we get $ch([e]) = tr(\frac{p}{2}dede) = tr(\frac{1}{8}pdpdp)$ since $de = dp/2$

Example 140. (Chern character on NCT) Since a lot of work has been done on the noncommutative torus (or $\mathcal{A}_\theta$ as we sometimes call it) I figure it is worthwhile to present at least one more related thing. This example is pulled from Loday.

The main purpose of this example is to show that the Chern character is in invariant that distinguishes between two algebras better than HC_* in some cases.

Recall that the Chern character is a pairing between cyclic cohomology and K_0. We need to compute these for $\mathcal{A}_\theta$. Luckily though they have been computed for us by Connes and Pimsner and Voiculescu. $K_0(\mathcal{A}_\theta) = \mathbb{Z} \oplus \mathbb{Z}$ generated by $\mathcal{A}_\theta$ and $\mathcal{S}(\mathbb{R})$. We've already seen how Schwartz functions are a right module over $\mathcal{A}_\theta$. Let us denote their classes by

$$\left\{ \begin{array}{ccc} [1] & \leftrightarrow & \mathcal{A}_\theta \\ [\mathcal{S}] & \leftrightarrow & \mathcal{S}(\mathbb{R}) \end{array} \right\}$$

We also know that $HC^2(\mathcal{A}_\theta) = \mathbb{C} \oplus \mathbb{C}$ with generators ϕ and $S\tau$. Then our pairings are

$$\left\{ \begin{array}{ll} \langle [1], S\tau \rangle = 1, & \langle [\mathcal{S}], S\tau \rangle = \theta, \\ \langle [1], \phi \rangle = 0, & \langle [\mathcal{S}], \phi \rangle = 1, \end{array} \right\}$$

What we see is that θ doesn't appear in HC^*, but does show up in ch.

5. What now?

There are several places to go from here. Among other things my main ideas as these

1. Several key ideas from noncommutative geometry have been discussed here. One of the main things I'm missing in these notes is anything related to deformation quantization. B. Fedosov has a book entitled *Deformation Quantization and Index Theory.* I would like to explore some of these ideas and see if they are applicable in any of the settings with which I am familiar.

2. Connes has a paper entitled *Cyclic Cohomology, Quantum Group Symmetries, and the Local Index Formula for* $SU_q(2)$. As far as I can tell, no one has yet extended this work to $\mathbb{C}_q[SU(n)]$ for $n > 2$. Of course this touches on another important subject related to NCG that I haven't really mentioned at all yet; quantum groups.

3. There are some interesting relationships between cyclic (co)homology and quantum field theory. B. Tsygan has suggested looking for a new proof that two specific cyclic cocycles are cohomologous using the language of quantum field theory.

4. There is a conjecture by a mathematician at Georgia Tech named Chris Heil. It is stated simply as this
   Let $f \in L^2(\mathbb{R})$ be nonzero then for any collection $(a_k, b_k) \in \mathbb{R}^2$ the set $\{f(t - a_k)e^{2\pi i b_k t}\}$ is linearly independent.

   Very little is known about the general case for this theorem. It has been shown to be true for any 2 points, any three points, and certain lattices. The case of lattices was shown using techniques of C^* algebras. I think some new headway can be made in the problem by using techniques of Pseudodifferential operators and also techniques related to the noncommutative torus.

Somewhere in these notes and the aforementioned ideas there should be a thesis lurking. My goal is to figure out exactly what to do with all of this and write a thesis.

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