THE HAAR FUNCTIONAL ON $SU_q(n)$

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Contents

1. Introduction

When attacking the problem of generalizing index theorems from low dimensional objects to higher dimensional analogous objects, one encounters many different ways of generalizing. Sadly, many of these generalizations lead one down a fruitless path. The following sections are the original work of the author in all cases where $n > 2$. It is the author's hope that these "correct" generalizations shed some light on how to compute with higher dimensional compact matrix quantum groups. The second section shows that the decomposition from classical representation theory yields a useful result in the quantum case. The third section uses in a specific way the decomposition of $SU_q(2)$ into direct summands indexed by pairs of integers to compute the Haar functional in the case $n = 2$. Section four shows that the techniques exploited in section three do not generalize to the higher dimensional cases without some arduous re-indexing. However, particular techniques from section three will come together in a rather fascinating way to give a Haar functional on $SU_q(n)$ that is strikingly similar to that on $SU_q(2)$ when the proper re-indexing has occured.

2. Peter-Weyl Type Decomposition of $SL_q(n)$

The Peter-Weyl type decomposition of $SL_q(n)$ is nearly identical to the classical case of $SL(n, \mathbb{C})$. For $n \in \mathbb{N}$ consider the algebras

$$\mathcal{O}(K_n) := \mathbb{C}[z_1, \ldots, z_n]/\langle z_1 \cdots z_n = 1\rangle.$$

These are the function algebras of the maximal tori of $SL(n, \mathbb{C})$. In the case $n = 2$ we have exactly the Laurent polynomials in one variable. One may, however put the structure of a Hopf algebra on $\mathcal{O}(K_n)$ by setting $\Delta(z_i) = z_j \otimes z_j$, $\epsilon(z_j) = 1$, $S(z_j) = z_j^{-1}$.

Remark 1. Since $\prod z_j = 1$ it is only necessary to use $n-1$ such z_j as

$$z_n = z_1^{-1} \cdots z_{n-1}^{-1}.$$

Now consider the homomorphisms

$$\phi : SL_q(n) \to \mathcal{O}(K_n)$$

given by

$$\phi(m_{i,j}^{(n)}) = \delta_{ij} z_j \tag{2.0.1}$$

These homomorphisms are in fact Hopf algebra homomorphisms. Now consider the homomorphisms

$$L_K : SL_q(n) \to \mathcal{O}(K_n) \otimes SL_q(n),$$
$$R_K : SL_q(n) \to SL_q(n) \otimes \mathcal{O}(K_n).$$

Given by

$$L_K = (\phi \otimes id) \circ \Delta$$
$$R_K = (id \otimes \phi) \circ \Delta$$

Now define the sets

$$\mathcal{A}[\alpha, \beta] := \{x \in SL_q(n) | L_K(x) = z^\alpha \otimes x, R_K(x) = x \otimes z^\beta\}. \tag{2.0.2}$$

Here, α and β are multi-indices and $z^\alpha = \prod_{i=1}^{n-1} z_i^{\alpha_i}$.

These sets $\mathcal{A}[\alpha, \beta]$ shall be known as the α-left, β-right invariant sets of $\mathcal{O}(SL_q(n))$. Elements of $\mathcal{A}[0, 0]$ shall be known as K *bi-invariant* elements.

The present goal is to show that these sets form a decomposition of $SL_q(n)$ and that the only elements which garner nontrivial Haar measure are those belonging to $\mathcal{A}[0, 0]$.

Presently, only $SL_q(2)$ shall receive attention. Once this decomposition is established for $n = 2$ the proper generalizations are easy to make. In the case of $SL_q(2)$ one sees that the sets $\mathcal{A}[m, n]$ are indexed by pairs of integers. Indeed the homomorphism ϕ acts by

$$a \mapsto z \qquad b \mapsto 0$$
$$c \mapsto 0 \qquad d \mapsto z^{-1}.$$

In order to check that these sets yield a decomposition one needs to check two things:

(1) All the generators fall into a single set.

(2) Multiplication of elements falls into a set i.e. if $x \in \mathcal{A}[m, n]$ and $y \in \mathcal{A}[r, s]$ then $xy \in \mathcal{A}[p, q]$ for some p, q.

Lemma 2. *In the case of* $SL_q(2)$ *one has*

(a) the generators a, b, c, d *belong to distinct sets, and*

(b) $\mathcal{A}[m, n] \cdot \mathcal{A}[r, s] \subset \mathcal{A}[m + r, n + s]$.

Proof. Part (a) is shown by direct computation. Only a will be shown here, the rest are done precisely the same.

$$\begin{aligned} L_K(a) = & \quad (\phi \otimes id)\Delta(a) \\ = & \quad (\phi \otimes id)(a \otimes a + b \otimes c) \\ = & \quad \phi(a) \otimes a + \phi(b) \otimes c \\ = & \quad z \otimes a \end{aligned}$$

$$\begin{aligned} R_K(a) = & \quad a \otimes \phi(a) + b \otimes \phi(c) \\ = & \quad a \otimes z \end{aligned}$$

Hence $a \in \mathcal{A}[1,1]$. Likewise $b \in \mathcal{A}[-1,1], c \in \mathcal{A}[1,-1], d \in \mathcal{A}[-1,-1]$.
As for (b) one needs to utilize the fact that ϕ, L_K, and R_K are homomorphisms. Let $x \in \mathcal{A}[m,n], y \in \mathcal{A}[r,s]$ then

$$\begin{aligned} L_K(xy) = L_K(x)L_K(y) = (z^m \otimes x)(z^r \otimes y) = z^{m+r} \otimes xy \\ R_K(xy) = R_K(x)R_K(y) = (x \otimes z^n)(y \otimes z^s) = xy \otimes z^{n+s} \end{aligned} \tag{2.0.3}$$

□

Therefore one may now write

$$\mathcal{O}(SL_q(2)) = \bigoplus_{m,n \in \mathbb{Z}} \mathcal{A}[m,n]. \tag{2.0.4}$$

When one attempts to replicate the proof for higher dimensions, there are few if any stopping blocks. In fact, the generators $m_{i,j}^{(n)}$ for $SL_q(n)$ all fall in the same way. Using the coproduct when $n > 2$ is marginally more annoying, but the homomorphisms kill off more elements than before. Furthermore, when checking the second condition, the only thing left to worry about is how to deal with multi-indices. This however gives no trouble in the actual computation. Therefore one may also write

$$\mathcal{O}(SL_q(n)) = \bigoplus_{m,n \in \mathbb{Z}^{n-1}} \mathcal{A}[m,n]. \tag{2.0.5}$$

What has happened is that the map ϕ sends $SL_q(n)$ into the coordinate algebra of the maximal torus of $SL(n,\mathbb{C})$ in direct analogy with the classical Peter-Weyl decomposition theorem.

Remark 3. Depending on the presentation of information shown to the reader, the generalization from $n = 2$ to $n > 2$ should be easy. However, there is one beautiful anomaly that occurs in the case $n = 2$. Namely one can show for every ℓ that

$$t^{\ell}_{i,j} \in \mathcal{A}[-2i,-2j] \tag{2.0.6}$$

where the $t^{\ell}_{i,j}$ are the matrix corepresentations from before. This is only possible because the indices of the decomposition are integers and not elements in an integer lattice. This particular piece of information is propitious when computing the Haar functional on $SU_q(2)$.

3. The Haar Functional on $SU_q(2)$

Woronowicz graced the mathematical world with a proof that there exists a unique bi-invariant linear functional satisfying

$$h(x) \cdot I = (id \otimes h)\Delta(x) = (h \otimes id)\Delta(x) \tag{3.0.7}$$

The existence and uniqueness alone are enough to show that $h(1) = 1$. Furthermore, in the case of $SL_q(2)$ one can easily determine $h(t^{\ell}_{i,j}) = 0$ when $\ell > 0$. One might wonder if there are any nontrivial nonvanishing elements under h. Indeed there are, but one needs to be clever to find them.

Lemma 4. *The only nonvanishing elements under h are the K bi-invariant elements.*

Proof. Let $x \in \mathcal{A}[m,n]$. Then using the bi-invariance of h and L_K, R_K one obtains $z^m h(x) = h(x) = h(x)z^n$. More explicitly one has

$$\begin{aligned} z^m h(x) = &\quad (id \otimes h)(z^m \otimes x) \\ = &\quad (id \otimes h)(\phi \otimes id)\Delta(x) \end{aligned}$$

But $(id \otimes h)$ and $(\phi \otimes id)$ commute so that

$$\begin{aligned} z^m h(x) = &\quad (id \otimes h)(\phi \otimes id)\Delta(x) \\ = &\quad (\phi \otimes id)(id \otimes h)\Delta(x) \\ = &\quad \phi(1)h(x) = h(x). \end{aligned}$$

One treats $h(x)z^n$ similarly. Therefore $h(x) = 0$ if $(m,n) \neq (0,0)$. □

Armed with this information, this first obvious choices to find a measure are ad and bc. Moreover in the case of $SU_q(2)$ one has an algebra equipped with a $*$-product and finds that

$$x \in \mathcal{A}[m,n] \iff x^* \in \mathcal{A}[-m,-n]. \tag{3.0.8}$$

This information becomes more prevalent in the higher dimensional cases. Another important piece of information to keep at bay is

$$x \in \mathcal{A}[m,n] \iff S(x) \in \mathcal{A}[-n,-m] \tag{3.0.9}$$

from which one may easily derive (3.0.8).

On $SU_q(2)$ the $*$-product yields $b^* = -qc$. Therefore the first element examined here will be $-qbc =: \zeta$. Utilizing Woronowicz's equations one finds

$$\begin{aligned} h(\zeta) = &\quad (id \otimes h) \circ \Delta(\zeta) \\ = &\quad (id \otimes h)(-q)(a \otimes b + b \otimes d)(c \otimes a + d \otimes c) \\ = &\quad -q(id \otimes h)(ac \otimes ba + ad \otimes bc + bc \otimes da + bd \otimes dc) \\ = &\quad adh(\zeta) + \zeta h(da) \\ = &\quad (1-\zeta)h(\zeta) + \zeta h(1 - q^{-2}\zeta) \\ &\quad \Longrightarrow h(\zeta) = \tfrac{1-q^{-2}}{1-q^{-4}} \end{aligned} \tag{3.0.10}$$

One important point to realize before going through further computations is that many elements vanish under h. It behooves one to project from $\mathcal{O}(SL_q(2))$ to $\mathcal{A}[0,0]$ before beginning any computations. Klimyk and Schmudgen have provided a few horrendous formulae for the general reader in this vain. Letting P be the aforemention projection; here they are:

$$(id \otimes P) \circ \Delta(\zeta^n) = \sum_{i+j=n} \left[\begin{array}{c} n \\ i \end{array} \right]^2_{q^{-2}} q^{2ij} \zeta^j (\zeta; q^2)_i \otimes \zeta^i (q^{-2}\zeta; q^{-2})_j \tag{3.0.11}$$

$$h(\zeta^n) \cdot I = \sum_{i+j=n} \left[\begin{array}{c} n \\ i \end{array} \right]^2_{q^{-2}} q^{2ij} \zeta^j (\zeta; q^2)_i h(\zeta^i (q^{-2}\zeta; q^{-2})_j) \tag{3.0.12}$$

$$h(\zeta^n) = \frac{1 - q^{-2n}}{1 - q^{-2(n+1)}} h(\zeta^{n-1}). \tag{3.0.13}$$

By noting the fact in $SL_q(2)$ that $\mathcal{A}[0,0] = \mathbb{C}[\zeta]$ one now knows how to compute the Haar functional within $SL_q(2)$. Moreover one knows how to compute $h(x^*x)$ and $h(xx^*)$ for any $x \in SU_q(2)$. The present goal then shall be to compute a more general Haar functional on $SU_q(2)$ using matrix corepresentations.

Consider the two Hermitian forms on $\mathcal{O}(SU_q(2))$ given by

$$\langle x, y \rangle_L = h(x^*y), \quad \langle x, y \rangle_R = h(xy^*), \quad x, y \in \mathcal{O}(SU_q(2)). \tag{3.0.14}$$

Since one should desire scalar products to be sesquilinear, a choice is necessary to determine which of these hermitian forms is an inner product. As presented here scalar products shall be linear in the first variable, making $\langle \cdot, \cdot \rangle_L$ and $\overline{\langle \cdot, \cdot \rangle}_R$ scalar products on the vector space $\mathcal{O}(SU_q(2))$.

Remark 5. Certain special properties of h and $\langle \cdot, \cdot \rangle$ necessitate comment. The Haar functional while linear, is not central. That is to say in general

$$h(xy) \neq h(yx). \tag{3.0.15}$$

Therefore one should like to have a method of interpolating between the two. The preferred method is to look for an automorphism ϑ such that

$$h(xy) = h(\vartheta(y)x). \tag{3.0.16}$$

It is here that one gets a glimpse of why $n = 2$ is so special. It this case one can solve ϑ directly on the generators and find that

$$\vartheta(a) = q^2 a, \qquad \vartheta(b) = b$$
$$\vartheta(c) = c, \quad \vartheta(d) = q^{-2} d$$

What is remarkable is that

$$\vartheta(x) = q^{m+n} x; \quad \forall x \in \mathcal{A}[m, n]. \tag{3.0.17}$$

And in particular at $n = 2$

$$\vartheta(t^{\ell}_{i,j}) = q^{-2(i+j)} t^{\ell}_{i,j}. \tag{3.0.18}$$

No such nicety is available for $n > 2$ as the indices m, n are points in an integer lattice rather than integers themselves. This problem will be solved later.

Two further remarks from definitions

(a) $\langle xz, y \rangle_R = \langle x, yz^* \rangle_R$ and similarly $\langle zx, y \rangle_L = \langle x, z^*y \rangle_L$

(b) $\langle x, y \rangle_L = \langle \vartheta(y), x \rangle_R$.

Theorem 6. *(i) The decomposition of $\mathcal{O}(SU_q(2))$ into matrix corepresentations is an orthogonal decomposition under $\langle\cdot,\cdot\rangle_L$ and $\langle\cdot,\cdot\rangle_R$*

(ii) The matrix corpresentations yield the following formulae for h.

$$\langle t^{\ell}_{i,j}, t^{\ell}_{i,j}\rangle_L = \frac{q^{-2i}}{[2\ell+1]_q} \tag{3.0.19}$$

$$\langle t^{\ell}_{i,j}, t^{\ell}_{i,j}\rangle_R = \frac{q^{2j}}{[2\ell+1]_q} \tag{3.0.20}$$

Proof. For part (i) it has already been established that $\langle t^{\ell}_{i,j}, t^{k}_{r,s}\rangle = 0$ if $(i,j) \neq (r,s)$. What is left to establish is orthogonality when $\ell \neq k$. This argument reduces to Schur's lemma for Hopf algebras.

Consider a $(2\ell+1)\times(2k+1)$ matrix M. Define $\tilde{M} := h(T^{\ell}MT^{k*})$ and $\tilde{M}' := h(T^{\ell*}MT^{k})$. Then $\tilde{M} = 0$ and $\tilde{M}' = 0$ when $\ell \neq k$. This assertion is shown by again considering the invariance properties of h.

$$\begin{aligned} T^{\ell}\tilde{M}T^{k*} &= (id\otimes h)((T^{\ell}\otimes I)(I\otimes T^{\ell})M(I\otimes T^{k*})(T^{k*}\otimes I)) \\ &= ((id\otimes h)\circ\Delta)(T^{\ell}MT^{k*}) \\ &= h(T^{\ell}MT^{k*}) = \tilde{M} \end{aligned}$$

Thus one obtains

$$T^{\ell}\tilde{M} = \tilde{M}T^{k}.$$

That is to say that $\tilde{M}$ intertwines irreducible corepresentations. By Schur's lemma the only invariant subspaces are empty or the whole space. Hence when $\ell \neq k$ $\tilde{M} = 0$. The same argument shows this for $\tilde{M}'$. Schur's lemma gives even more information however. Not only is the invariant subspace for $\tilde{M}$ the whole space, but $\tilde{M}$ and $\tilde{M}'$ take the special forms

$$\tilde{M} = \alpha I, \quad \tilde{M}' = \alpha' I \quad \alpha, \alpha' \in \mathbb{C}.$$

The quantities one now seeks are $\langle t^{\ell}_{i,j}, t\rangle_L = \alpha'_i$ and $\langle t^{\ell}_{i,j}, t\rangle_R = \alpha_j$. But one already has a relation between these two numbers in the guise of

$$\langle t^{\ell}_{i,j}, t^{\ell}_{i,j}\rangle_L = \langle \vartheta(t^{\ell}_{i,j}), t^{\ell}_{i,j}\rangle_R.$$

Hence

$$\alpha'_i = q^{-2(i+j)}\alpha_j. \tag{3.0.21}$$

Moreover there exists α so that $\alpha = q^{2i}\alpha'_i = q^{-2j}\alpha_j$ for all i,j. However, from the computation above

$$\begin{aligned} h(\zeta^{2\ell}) &= h((b^*b)^{2\ell}) \\ &= \langle t^{\ell}_{i,-\ell}, t^{\ell}_{i,-\ell}\rangle \\ &= \alpha_{-\ell} = \frac{q^{-4\ell}(1-q^{-2})}{1-q^{-4\ell-2}} \end{aligned} \tag{3.0.22}$$

Therefore one obtains

$$\alpha = \frac{q^{-2\ell}(1-q^{-2})}{1-q^{-4\ell-2}}$$

and

$$\alpha_j = \frac{q^{2j}}{[2\ell+1]_q},$$
$$\alpha'_i = \frac{q^{-2i}}{[2\ell+1]_q} \tag{3.0.23}$$

□

4. Generalizing to $SU_q(n)$

One of the many conveniences ascribed to the case $n = 2$ is the fact that the automorphism ϑ may be written $\vartheta(x) = q^{m+n}x$ when $x \in \mathcal{A}[m,n]$. Perhaps one of the first steps in generalizing to the $n > 2$ case should be to produce a similar automorphism that accounts for the noncommutative property of h. One should like to have

$$h(xy) = h(\vartheta(y)x) \quad \forall x, y \in \mathcal{O}(SL_q(n)). \tag{4.0.24}$$

The first issue encountered here is that $\mathcal{O}(SU_q(n))$ cannot be reduced to n generators as in the case $n = 2$. In fact, since the $*$-structure in $SU_q(n)$ involves quantum determinants of cofactors $\mathcal{O}(SU_q(n))$ properly has n^2 generators. With this in mind, the desired automorphism ϑ requires n^2 parameters to be fully determined. There are only a handful of properties that one can guarantee of ϑ, namely

(1) If $\vartheta(x) = \beta x$ then $\vartheta(x^*) = \beta^{-1}x^*$. This insures that the determinant relations hold on $\mathcal{O}(SU_q(n))$

(2) When $x^*x = xx^*$ then $\vartheta(x) = x$. Specifically this happens at $x = t^\ell_{-\ell,\ell}$ and $x = t^\ell_{\ell,-\ell}$. Note that when $n \neq 2$ then ℓ does not increment by $1/2$, but rather by $\begin{pmatrix} n+k-1 \\ n-1 \end{pmatrix}$ halves at the kth step.

The form $\vartheta(t^\ell_{i,j}) = q^{-2(i+j)}t^\ell_{i,j}$ from $\mathcal{O}(SU_q(2))$ fortunately yields an acceptable automorphism in the higher cases. What one needs to check in this case is that this particular automorphism coincides with commutation relations on $SU_q(n)$.

Example 7. Consider the following necessities of h and their correlations with relations on $SU_q(n)$.

$$\begin{aligned}
\sum_{j=1}^{n} m_{1,j}m^*_{1,j} = 1, &\qquad \sum_{j=1}^{n} q^{-2j}m^*_{1,j}m_{1,j} = 1 \\
\sum_{i=1}^{n} m^*_{i,1}m_{i,1}, &\qquad \sum_{i=1}^{n} q^{2i}m_{i,1}m^*_{i,1} \\
h(\sum_{j=1}^{n} m_{1,j}m^*_{1,j}) &= \sum_{j=1}^{n} h(m_{1,j}m^*_{1,j}) = 1 \\
h(\sum_{j=1}^{n} q^{-2j}m^*_{1,j}m_{1,j}) &= \sum_{j=1}^{n} q^{-2j}h(m^*_{1,j}m_{1,j}) = 1
\end{aligned} \tag{4.0.25}$$

This seems to suggest that h varies directly with the sub-indices of the generators. Fortunately this is the case when $n = 2$. Another important clue derived from these equations is that when using the left or right invariance of h the coproducts will yield unsightly equations involving scalars hitting elements of the algebra which have specific relations. For example when trying to compute $h(m_{1,n}m_{1,n}^*)$ one arrives at

$$h(m_{1,n}m_{1,n}^*)\cdot I=\sum_{j=1}^{n}m_{1,j}m_{1,j}^*h(m_{j,n}m_{j,n}^*) \tag{4.0.26}$$

Clearly it is the case that $h(m_{1,n}m_{1,n}^*)\neq 0$ so one must account for the fact that $\sum_j m_{1,j}m_{1,j}^*=1$. What one must conclude is that $h(m_{1,n}m_{1,n}^*)=h(m_{j,n}m_{j,n}^*)$ for every $j\in\{1,\dots,n\}$.

In a similar way, one can play all the tricks in computing relations between $h(m_{i,j}m_{i,j}^*)$ and $h(m_{i,j}^*m_{i,j})$. The relations can be listed as follows:

(1) $h(m_{i,j}m_{i,j}^*)=\langle m_{i,j},m_{i,j}\rangle_R$ is constant in j

(2) $h(m_{i,j}^*m_{i,j})=\langle m_{i,j},m_{i,j}\rangle_L$ is constant in i

It is now convenient to move into computations with matrix corepresentations. Note $m_{i,j}=t^1_{i-2,j-2}$. Here one should like to have the automorphism ϑ in hand. Then one needs to check $\vartheta(t^\ell_{i,j})=q^kt^\ell_{i,j}$ against the given relations on $h(m_{i,j}m_{i,j}^*)$. One will see after a short computation

$$\vartheta(t^\ell_{i,j})=q^{-2(i+j)}t^\ell_{i,j}. \tag{4.0.27}$$

This is exactly the form of ϑ from $n=2$. Then using the invariance of h one finds

$$h(t^\ell_{i,\ell}t^{\ell*}_{i,\ell})=\sum_{k=-\ell}^{\ell}h(t^\ell_{i,k}t^{\ell*}_{i,k})t^\ell_{k,\ell}t^{\ell*}_{k,\ell} \tag{4.0.28}$$

Putting all the steps together one needs to use ϑ, the respective constancy conditions in i and j, the quantum determinant relations on $\mathcal{O}(SU_q(n))$, and the bivariance of h to arrive at

$$\langle t^\ell_{i,j},t^\ell_{i,j}\rangle_R=\frac{q^{2j}}{\sum_{k=-\ell}^{\ell}q^{2k}}=\frac{q^{2j}}{[2\ell+1]_q} \tag{4.0.29}$$

$$\langle t^\ell_{i,j},t^\ell_{i,j}\rangle_L=\frac{q^{-2i}}{\sum_{k=-\ell}^{\ell}q^{2k}}=\frac{q^{2i}}{[2\ell+1]_q}. \tag{4.0.30}$$

(4.0.31)

These are the desired formulae for h in $\mathcal{O}(SU_q(2))$ for any n. The difference in the higher dimensional cases is simply the indexing on ℓ. Rather elementary combinatorics come into play to aid one in the discovery that successive representations of $SU_q(n)$ need not exist for each half integer.

References

[1] Clark Alexander *Matrix Corepresentations of $SL_q(n)$ and $SU_q(n)$* In progress 2007 http://www.math.northwestern.edu/∼clarkbar/papers.html.

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