OPERADIC CORRESPONDENCE OF COMMUTATIVE, ASSOCIATIVE, AND LIE ALGEBRAS

CLARK ALEXANDER

There are two questions at hand. The first is to describe the correspondence of making an associative algebra from a commutative associative algebra. The second is the describe the correspondence of making a Lie algebra from an associative algebra.

Let's begin with the first statement. At the level of categories this is almost a trivial statement. Associative commutative algebras are simply a subcategory in associative algebras. The way to produce an associative algebra from a commutative algebra is to simply forget the structure of commutativity and only consider the associativity. This is almost too simple. At the level of operads we have a little more work to do.

1. *COMM* and *ASSOC*

Recall that we have the following descriptions for the operads $COMM$ and $ASSOC$.

$$COMM(n) := \{(k\langle x_1 \cdots x_m\rangle, \circ_j\} \tag{1.0.1}$$

Where $\langle x_1 \cdots x_n\rangle$ is a monomial and $\circ_j$ is defined by

$$COMM(n) \otimes COMM(m) \xrightarrow{\circ_j} COMM(n+m-1)$$

on representatives given explicitly by

$$\langle x_1 \cdots x_n\rangle \circ_j \langle y_1 \cdots y_m\rangle = \langle x_1 \cdots x_{j-1}(y_1 \cdots y_m)x_{j+1} \cdots x_n\rangle \tag{1.0.2}$$

Similarly on $ASSOC$ we have

$$ASSOC(n) := \{k[S_n], \circ_j\} \tag{1.0.3}$$

and $\circ_j$ is defined exactly the same way on $ASSOC$. Now we notice that $dim_k(COMM(n)) = 1$ and $dim_k(ASSOC(n)) = n!$ for any n.

The generators of $ASSOC(n)$ are written

$$\langle x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}\rangle$$

where $\sigma \in S_n$ and we use σ^{-1} to preserve group actions. Let us give $\circ_j$ explicitly for $ASSOC$.

$$\langle x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}\rangle \circ_j \langle y_{\tau^{-1}(1)} \cdots y_{\tau^{-1}(m)}\rangle$$
$$= \langle x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(j-1)}(y_{\tau^{-1}(1)} \cdots y_{\tau^{-1}(m)})x_{\sigma^{-1}(j+1)} \cdots x_{\sigma^{-1}(n)}\rangle$$

We write this relation $\sigma \circ_j \tau \in S_{n+m-1}$ when $\sigma \in S_n$ and $\tau \in S_m$.

Notice now on operads we are actually looking for a map

$$ASSOC \longrightarrow COMM.$$

With our given structure we are ready to define.

$$ASSOC(n) \ni \langle x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)} \rangle \mapsto \langle x_1 \cdots x_n \rangle \in COMM(n) \tag{1.0.4}$$

and extend by linearity. This map is of course degenerate. We map the $n!$ generators directly to the one generator in $COMM(n)$. This will become important when we consider the map

$$LIE \longrightarrow ASSOC.$$

For now we have shown

$$ASSOC \twoheadrightarrow COMM.$$

2. *ASSOC* and *LIE*

The problem of producing a Lie algebra from an associative algebra is also fairly simple at the level of categories. We simply impose a bracket on an associative algebra and require

$$[x, y] = xy - yx$$

or something related (we can change sign). We can see that this is indeed a Lie algebra since our bracket satisfies the Jacobi identity by virtue of associativity. That is to say when we consider the 12 term thing we get by expanding

$$[[x, y], z] + [[z, x], y] + [[y, z], x]$$

we find cancelations by virtue of associativity.

Let's move onto the problem of defining a map

$$LIE \longrightarrow ASSOC.$$

Our presentation of LIE is

$$LIE(n) := \{k\langle[x_1 \cdots x_n]\rangle, \circ_j\} \tag{2.0.5}$$

Here we write angled brackets and square brackets to remind us that we are considering a Lie algebra structure. For example $LIE(2)$ is one-dimensional and this corresponds to the sign representation of S_2. The product is again $\circ_j$ given by substitution

$$[x_1 \cdots x_n] \circ_j [y_1 \cdots y_m] = [x_1 \cdots x_{j-1}[y_1 \cdots y_m]x_{j+1} \cdots x_n]$$

Let us note quickly that the generators of $LIE(n)$ are the monomials $[x_1 \cdots x_n]$ where this product is considered to mean *nested* brackets.

Our correspondence for LIE and $ASSOC$ looks like this

$$LIE(n) \ni \mathfrak{g} = k\langle[x_1 \cdots x_n]\rangle \hookrightarrow k\langle x_1 \cdots x_n\rangle = U(\mathfrak{g}) \in ASSOC(n) \tag{2.0.6}$$

This essentially tells us in categories to take the universal enveloping algebra of a Lie algebra. by construction this is an associative algebra.

Notice what we have done at this point. We've produced two maps

(1) $LIE \hookrightarrow ASSOC$

(2) $ASSOC \twoheadrightarrow COMM$

Combining these we get a short exact sequence

$$0 \to LIE \to ASSOC \to COMM \to 0. \tag{2.0.7}$$

This tells us something important.

Proposition 1. *The only trivial Lie algebras are commutative algebras. Conversely if we begin with a commutative algebra and impose a bracket on it, we end up with a trivial Lie algebra.*

Let's look at a quick example for $n = 2$

$$\begin{array}{ccccc} LIE(2) & \longrightarrow & ASSOC(2) & \longrightarrow & COMM(2) \\ \updownarrow & & \updownarrow & & \updownarrow \\ [x_1 x_2] & \mapsto & x_1x_2 - x_2x_1 & \mapsto & x_1x_2 - x_1x_2 = 0 \end{array}$$

Notice in the second map $x_2x_1 \mapsto x_1x_2$ as all the generators for $ASSOC(2)$ map to *the* generator for $COMM(2)$.

E-mail address: clarkbar@math.northwestern.edu