A SHORT NOTE ON LINEAR DIFFERENTIAL EQUATIONS

CLARK ALEXANDER

Contents

Introduction

This is a short note intended to show a few techniques used to study the theory of linear differential equations. Most techniques are not computationally efficient, but the solutions are indeed correct. There are a myriad of topics which are conspicuously absent. Most notably singular equations and series solutions near singularities are absent. As we will see it is not necessary to amend our solution to accommodate differential equations which have repeated eigenvalues, nor will it be necessary to amend it for complex eigenvalues. The final solution to a system of first order linear equations will be valid in neighborhoods of regular points.

We'll begin the actual discussion by solving the most basic first order equations in one dimension as a basis for interpolating to higher dimensions. The main tool for the solution of systems of linear equations will be exponentiating matrices. This will be discussed in some detail in section two. After we have established this tool it is a short jump into solving *all* linear differential equations in $\mathbb{R}^n$ for *all* n (away from singular points). We will make a few small departures from the main task to work on differential equations via other special techniques. These sections are added purely for the interest of the author. This note concludes with a brief stint in the land of PDE's. Of course the solution technique exhibited in the final section involves exponentiating a differential operator.

Even if this notes do not turn out to be useful, it is the author's hope that they are at least interesting.

1. Linear First Order ODE in One Dimension

First order differential equations in one dimension are markedly simple. They come in two basic varieties. Letting $\dot{y} := \frac{dy}{dt}$ they are

(1) $\dot{y} = ay$ with $a \in \mathbb{R}$ and

(2) $\dot{y} + p(t)y = q(t)$ with $p, q \in C(\mathbb{R})$

The first type can be solved using elementary integral calculus as follows.

$$\frac{dy}{dt} = ay \implies \frac{1}{y}\frac{dy}{dt} = a$$

or as we usually write by abuse of notation

$$\frac{dy}{y} = adt$$

then we integrate both sides (with respect to t) to obtain

$$ln(y) = at + c \implies y(t) = Ce^{at} \tag{1.0.1}$$

and $C = e^c = y(0)$. One important thing to note about differential equations of any order is that the number of initial conditions needs to equal the order of the equation to ensure a unique solution. For the remainder of these notes we'll let our initial conditions be dictated from $t = 0$, which will be denoted $y(0), y_0, \mathbf{x}(0)$, or $\mathbf{x}_0$.

We have arrived at the solution to the first type of equation given by

$$y(t) = e^{at}y(0) \tag{1.0.2}$$

It is easy to check that this solution holds by simply taking derivatives.

Moving on to the second basic equation we use a special technique called an *integrating factor* usually denoted μ. To solve

$$\dot{y} + p(t)y = q(t)$$

we multiply through by μ to obtain

$$\mu\dot{y} + \mu py = \mu q \tag{1.0.3}$$

The idea is to make the left hand side look like the derivative according to the product rule. That is

$$\frac{d}{dt}(\mu y) = \frac{d}{dt}(\mu)y + \mu\frac{d}{dt}(y) = \mu y + \mu py \tag{1.0.4}$$

Collecting terms and simplifying we get

$$\dot{\mu} = p\mu$$

which we know how to solve as above.

$$\mu(t) = e^{\int_0^t p(s)ds}\mu(0). \tag{1.0.5}$$

Coming back to the original equation we have

$$\frac{d}{dt}(\mu y) = \mu q. \tag{1.0.6}$$

Integrating both sides again with respect to t we arrive at

$$y(t) = \frac{1}{\mu(t)}\int_0^t \mu(s)q(s)ds \tag{1.0.7}$$

with $\mu(t)$ as above.

2. Exponentiating Things

This section will comprise the bulk of the important information for this note. We begin by recalling the formula

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2.0.8}$$

And furthermore that this series has an infinite radius of convergence. If we wanted to consider this type of function on a space other than real (or complex) numbers we need to ask what properties we should require of x. As it is, we only need x so that x^n makes sense for all n. In this way we can make sense of e^A when A is an operator or a matrix. The one small detail we need to pay attention to is that 1 becomes I or the appropriate unit for our object A. Given this we have a new equation to consider.

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \tag{2.0.9}$$

We can guarantee that this converges as e^x has infinite radius of convergence. All we need to do is consider an inequality such as

$$\|e^A\| \leq e^{\|A\|}.$$

Here we can make $\|A\|$ as large as possible and $e^{\|A\|}$ still converges.

The real problem with exponentiating things is that e^A is hard to compute. Even for real numbers we have a hard time saying exactly what e^x is. Let's do some examples.

Example 1. When A is a square matrix we have a real problem computing A^n unless A is diagonalizable or nilpotent. The first easy example is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Of course in this case $A^2 = 0$ so our entire series is

$$e^A = I + A + 0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In most cases when solving differential equations we'll be concerned with e^{tA} and not simply e^A so in this case

$$e^{tA} = I + tA + 0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

In general if A is diagonalizable or has a full eigenspace then we can compute e^{tA} as follows. Suppose A has a decomposition

$$AP = PD$$

Where P is a matrix of eigenvectors (with determinate 1) and D is a diagonal matrix with the corresponding eigenvalue on the diagonal. Diagrammatically we have

$$A[v_1 \cdots v_n] = [v_1 \cdots v_n] \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} \tag{2.0.10}$$

So that $Av_i = \lambda_i v_i$. Now if A has a full eigenspace then P is invertible and we have

$$A = PDP^{-1}. \tag{2.0.11}$$

Now by plugging in PDP^{-1} into our formula for e^A we arrive at

$$e^A = e^{PDP^{-1}} = \sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!} = I + PDP^{-1} + \frac{1}{2} PDP^{-1}PDP^{-1} + \cdots$$

But of course matrix multiplication is associative so that

$$PDP^{-1}PDP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

And computing powers of diagonal matrices is as simple as multiplying real numbers

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} \implies D^j = \begin{pmatrix} \lambda_1^j & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^j \end{pmatrix}$$

And thus we see that

$$e^D = \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n} \end{pmatrix}$$

So we end up with the formula

$$e^A = e^{PDP^{-1}} = P(\sum_{k=0}^{\infty} \frac{D^k}{k!})P^{-1} = Pe^DP^{-1} \tag{2.0.12}$$

Of course sometimes we can simply find a pattern for A^n when A is simple enough.

Example 2. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $A^4 = I$. The more interesting computation is e^{At} so we'll work on that one. Since we have the following relations $A^2 = -I, A^3 = -A, A^4 = I$ our power series becomes significantly easier.

$$e^{At} = I + At - \frac{It^2}{2} - \frac{At^3}{6} + \frac{It^4}{24} + \cdots$$

Collecting terms we arrive at

$$e^{At} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} & \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \\ -\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \end{pmatrix}$$

Which we can recognize as a rotation matrix

$$e^{At} = \begin{pmatrix} cos(t) & sint(t) \\ -sin(t) & cos(t) \end{pmatrix}.$$

Now we come to the difficult part: as we know, matrices in general do not commute. And since e^{At} is a matrix we might not expect it to commute with any other given matrix. The problem arises when we consider matrices such as $e^{(A+B)t}$. Our natural instinct is to write $e^{(A+B)t} = e^{At}e^{Bt}$. But what if we wrote $e^{(A+B)t} = e^{(B+A)t} = e^{Bt}e^{At}$? We would then be presupposing that $e^{At}e^{Bt} = e^{Bt}e^{At}$. But for any two general matrices this is not the case. In fact we have this particular relation only when $[A, B] = 0$ (i.e. $AB = BA$). In fact trying to compute $e^{(A+B)t}$ in terms of e^{At} and e^{Bt} becomes a fairly horrible process.

Using the same idea of power to exponentiate matrices we can also take logarithms. Just to demonstrate the awful nature of $e^{At}e^{Bt}$ we consider the following matrix Z.

$$Z = log(e^{At}e^{Bt}).$$

By expanding $e^{(A+B)t}$ in a power series and collecting terms we arrive at

$$Z = z_1 + z_2 + z_3 + \cdots$$

where the z_i are entries in the so-called *Dyson series* from quantum mechanics. These z_i are given as follows (for the first few)

(1) $z_1 = At + Bt$

(2) $z_2 = \frac{t^2}{2}[A, B]$

(3) $z_3 = \frac{t^3}{12}([A, [A, B]] - [B, [A, B]])$

This long digression is essentially to remind us that matrices do not function like numbers no matter how hard we try. However the silver lining to this whole non-commutative debacle is that e^A is always in invertible matrix with inverse e^{-A}. Of course we know that $[A, -A] = 0$ so that $e^Ae^{-A} = e^{A-A} = e^0$. And $e^0 = I$ just like in the case of numbers.

Now that we have a good handle on how bad exponentiating matrices truly can be let's look at exponentiating operators. Again we need to make sure that taking powers of the operator makes sense. Essentially we need operators T such $T : \mathcal{H} \to \mathcal{H}$ where $\mathcal{H}$ is any sort of nice algebra. This could be a Hilbert or pre-Hilbert space, a C^* algebra or pre-C^* algebra or plenty of other things. For the sake of interest of the author we're going to look at one particular example. Consider exponentiating $s\frac{d}{dx}$ acting on $C^\infty(\mathbb{R})$ with $s \in \mathbb{R}$.

Proposition 3. *When considering* $s\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ *for* $s \in \mathbb{R}$ *we have the following formula.*

$$e^{s\frac{d}{dx}}f(x) = f(x + s) \tag{2.0.13}$$

In essence $e^{s\frac{d}{dx}}$ *is a shifting operator.*

Proof. Recall that the Taylor series for a function f around a point a is given by

$$f(y) = f(a) + f'(a)(y - a) + \frac{1}{2}f''(a)(y - a)^2 + \cdots$$

Certainly we can rewrite this as

$$f(y) = 1f(a) + (y - a)\frac{d}{dy}|_a f + \frac{(y - a)^2}{2}(\frac{d}{dy})^2|_a f + \cdots$$

By factoring on the left we can again rewrite this as

$$f(y) = (1 + (y - a)\frac{d}{dy} + \frac{(y - a)^2}{2}(\frac{d}{dy})^2)|_a f$$

We can play this game again and write

$$f(y) = (\sum_{k=0}^{\infty} \frac{(y - a)^k}{k!}(\frac{d}{dy})^k)|_a f$$

which looks conspicuously like

$$f(y) = e^{(y-a)\frac{d}{dy}} f.$$

So by letting $y = x + s$ and $a = x$ recover our formula as desired. $\square$

We can extend this technique to other differential operators such as the Laplacian. We can also use the technique of diagonalization to compute exponentials of operators. For example the Laplacian Δ has a full set of eigenvectors.

$$\Delta\phi_j = \lambda_j \phi_j$$

and these $\{\phi_j\}_{j=0}^{\infty}$ form a basis for $L^2(\mathbb{R}^n)$. So in this case we can essentially write $f = \sum_j c_j \phi_j$ and

$$e^{\Delta} f = e^{\Delta}(\sum_j c_j \phi_j) = \sum_j c_j e^{\lambda_j} \phi_j \tag{2.0.14}$$

This is the spectral (or functional) calculus and is generally how we compute functions of operators. The general form looks like

$$f(T)\phi_j = f(\lambda_j)\phi_j \tag{2.0.15}$$

where $T\phi_j = \lambda_j \phi_j$. We only need to make sure that f is 'nice enough.'

3. An Example of a Delay Differential Equation

As per the computation above we have the formula

$$e^{s\frac{d}{dt}} f(t) = f(t+s)$$

when f has a Taylor series with radius of convergence $\geq s$. In any case let us reconsider our first example again.

$$\frac{dy}{dt} = ay$$

where $a \in \mathbb{R}$. We should be able to see directly from spectral calculus that $e^{\frac{d}{dt}} y = e^a y$ when $\dot{y} = ay$. If this is not apparent just expand $e^{\frac{d}{dt}} y$ in a formal power series again and use the differential equation. Furthermore have have the string of equalities

$$e^{as} y(t) = e^{s\frac{d}{dt}} y(t) = y(t+s) \quad \forall t, s \tag{3.0.16}$$

Now we set $t = 0$ and arrive at

$$e^{as} y(0) = y(s) \quad \forall s \tag{3.0.17}$$

as we saw before using simple integral calculus.

4. Systems of First Order ODE

Very much like first order ODE in one dimension systems of first order ODE come in two basic varieties.

(1) $\frac{d}{dt}\mathbf{x} = A(t)\mathbf{x}$ with $A(t) \in M_n(C(\mathbb{R}))$ and

(2) $\frac{d}{dt}\mathbf{x} + A(t)\mathbf{x} = \mathbf{b}(t)$ with $A(t)$ as above and $\mathbf{b}(t) \in C(\mathbb{R})^n$

And fortunately for us the solutions are completely analogous to those in §1. The only thing we need to know to check our solutions is that derivatives of vectors and matrices are taken component wise. With that in mind the general solution to a first order system of differential equations is given by

$$\mathbf{x}(t) = e^{-\int_0^t A(s)ds}(\mathbf{x}_0 + \int_0^t e^{\int_0^\tau A(s)ds} b(\tau)d\tau) \tag{4.0.18}$$

Note that we have rearranged the first type of equation to be

$$\frac{d}{dt}\mathbf{x} + A(t)\mathbf{x} = 0$$

so that all first order systems now look like the second type of equation. It is easy to check that this solution indeed solves our differential equations. From here we have only a short jump to make to solve all orders of differential equations.

5. Converting Higher Order Equations into First Order Equations

In a normal course (one quarter or one semester) the next topic after first order equations would naturally be second order equations. Usually classes tackle this topic using homogeneous equations with constant coefficients and then showing a hundred special techniques for solving them when the roots are repeated and then even more for the nonhomogeneous case. The usual suspects show up in this form:

$$\frac{d^2}{dt^2}y + a\frac{d}{dt}y + by = 0.$$

By taking a huge hint from the first order homogeneous systems we guess that $y(t) = e^{rt}$ and then proceed with taking derivatives to arrive at the *characteristic equation*

$$(r^2 + ar + b)e^{rt} = 0 \tag{5.0.19}$$

which is simply a quadratic equation and thus easily solvable. Of course there are usually two solutions to this equation called r_1, r_2 so that our final solution should look something like

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Then we have to deal with $r_1 = r_2, r_1 = \bar{r}_2 \in \mathbb{C}$ etc. It gets horrible and that's only the homogeneous case. Let's solve it a better way by using a clever trick to convert our second order equation into a system of first order equations.

The form of differential equation we know how to solve involves vectors and only one derivative. In order to transform our current second order equation into a nice first order one we'll have to introduce a new variable called $z(t)$.

Let's consider our general second order equation right off the bat as

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \tag{5.0.20}$$

Then we define our new variable

$$z(t) := \frac{d}{dt}y(t). \tag{5.0.21}$$

with this we have two differential equations that look like

$$\dot{y} = z \tag{5.0.22}$$
$$\dot{z} = -qy - pz + g \tag{5.0.23}$$

Which in matrix form looks like

$$\frac{d}{dt}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \tag{5.0.24}$$

And now certainly by rewriting

$$\mathbf{x}(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$$

and

$$-A(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}$$

$$\mathbf{b}(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

we recover a first order system of differential equations.

The power of this solution is that not only do we solve for $y(t)$, but also for $\dot{y}(t)$ simultaneously. Our conundrum with initial conditions is also solved. Instead of needing two initial condition, we only need one. However it is important to note that the two conditions (i.e. y_0 and y_0') now become $\mathbf{x}_0$.

This technique easily extends to third and fourth order equations and beyond. For the sake of argument let's give the matrix $A(t)$ for a third order equation.

$$y''' + p(t)y'' + q(t)y' + r(t)y = g(t)$$

becomes

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{pmatrix}\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ g(t) \end{pmatrix} \tag{5.0.25}$$

The only thing left to cover is systems of higher order equations. Again the idea is to define enough new variables in order to make our system of higher order equations reduce to a system of first order equations, which is always possible. This essentially covers all of linear differential equations (without singularities).

6. Solving the Heat Equation Using Spectral Calculus

This section is a bit a departure from the ODE's we've been solving. Here we will see a technique from functional analysis using the so-called *solution operator* to solve the heat equation.

Recall that the heat equation in n dimensions is given as follows

$$\frac{du}{dt} = \Delta u$$

Where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ and $u = u(x,t)$ with $x \in \mathbb{R}^n$.

We begin the solution by introducing the *solution operator* given by $e^{-t\Delta}$.

Theorem 4. *The solution to the heat equation is given by*

$$u(x,t) = e^{t\Delta} f(x) \tag{6.0.26}$$

where $f(x) \in C^\infty(\mathbb{R}^n)$

Proof. Consider solving the equation

$$\frac{d}{dt}(e^{-t\Delta}u) = 0 \tag{6.0.27}$$

We get the following

$$\begin{aligned} 0 &= \frac{d}{dt}(e^{-t\Delta}u) \\ &= (-\Delta)(e^{-t\Delta}u) + (e^{-t\Delta}\frac{du}{dt}). \end{aligned} \tag{6.0.28}$$

Since Δ commutes with $e^{-t\Delta}$ we can rearrange the above equation to see

$$e^{-t\Delta}(-\Delta u + \frac{du}{dt}) = 0 \tag{6.0.29}$$

and since $e^{-t\Delta}$ is invertible with inverse $e^{t\Delta}$ we can rearrange once more to obtain

$$\frac{du}{dt} = \Delta u$$

as desired. Now the solution boils down to finding functions f so that $\frac{df}{dt} = 0$. Of course all functions f not depending on t satisfy this property. Therefore we have reduced the entire problem to solving

$$(e^{-t\Delta}u) = f(x). \tag{6.0.30}$$

Again since $e^{-t\Delta}$ is invertible with inverse $e^{t\Delta}$our final solution is

$$u(x,t) = e^{t\Delta} f(x).$$

We only require $f \in C^\infty$ so that hitting it with $e^{t\Delta}$ makes sense. □

We can also see that this technique will extend to all differential equations of the form

$$\frac{du}{dx_j} = D(x_1, \ldots, \hat{x}_j, \ldots, x_n)u$$

where $\hat{x}_j$ means that x_j is removed. In other words this technique extends to differential equations where D is a differential operator not depending on an isolated x_j.

E-mail address: `clarkbar@math.northwestern.edu`