Two-tiered electoral systems

Standard example: Voters elect electors who in turn elect president.

In US, there are 51 electoral districts—the states and the District of Columbia.

Except for Nebraska and Maine, each assigns all its votes to the candidate who gets a plurality of the popular vote in the district.

Is Poland crazy?

Council of Ministers of European Community is a two-tiered system

Each member elects a minister. Minsters have different voting weights—depending on their country’s population.

Germany says the voting weight should be proportional to population.

Poland says no, it should be proportional to the square root of the population.

What's Poland's angle here?

Populations: Germany 82 million.
Poland 38 million

$$\frac{82}{38} = 2.16 \quad \sqrt{\frac{82}{38}} = 1.47$$

So, should Germany's weight be $2.16\times$ Poland's or $1.47\times$?

Penrose-Banzhaf model

Lionel Penrose (1898–1972) (Geneticist, father of Roger Penrose) in 1940's proposed the square root rule:

The voting weight of a representative should be proportional to the square root of the population represented.

The Banzhaf model

John Banzhaf III (Law professor, gadfly http://banzhaf.net) discovered in the late 1950s:

Nassau County, Long Island was unjustly governed (but didn't know).

Some members of the county's Board of Supervisors were dummy voters, in the sense that there votes on legislation did not count.

Banzhaf index

When a vote is recorded, a **critical voter** is one who would change they outcome if he or she alone switched sides.

A **voting combination** is a possible record of the votes. With N voters, there are 2^N voting combinations.

A voter's **Banzhaf index** is the number of voting combinations in which he/she/it is a critical voter.

The Nassau county dummies had zero Banzhaf indices. Two supervisors had weight 9, one had weight 7, there was a weight 3, and two 1's. To approve a bill, 16 votes were necessary.

Calculating the Banzhaf index

Assume N voters, weights $w_1, w_2, \ldots, w_N$, quota to win is q.
Let

$$P_j(x) = \prod_{m \neq j} (1 + x^{w_m}), \text{ and } M = \sum_{m=1}^{N} w_m.$$

$$P_j(x) = 1 + a_1 x + \cdots + a_{M-w_j} x^{M-w_j}$$

$a_p =$ number of sets of voters not including voter j that have total weight p. Thus, the
Banzhaf index of voter $j = 2 \sum_{p=q-w_j}^{q-1} a_p$.

Example: Nassau County

$M = 9 + 9 + 7 + 3 + 1 + 1 = 30$, $q = 16$.

$$\begin{aligned} P_1 &= (1+x^9)(1+x^7)(1+x^3)(1+x)^2 \\ &= x^{21} + 2x^{20} + x^{19} + x^{18} + 2x^{17} + x^{16} + x^{14} \\ &\quad + 2x^{13} + 2x^{12} + 3x^{11} + 3x^{10} + 2x^9 + 2x^8 \\ &\quad + x^7 + x^5 + 2x^4 + x^3 + x^2 + 2x + 1 \end{aligned}$$

Banzhaf index of weight-9 voter is

$$2\sum_{p=7}^{15} a_p = 2(1+2+2+3+3+2+2+1) = 16$$

The dummy from Oyster Bay

This supervisor had weight 3. His polynomial was

$$\begin{aligned} P_4 &= (1+x^9)^2(1+x^7)(1+x)^2 \\ &= x^{27}+2x^{26}+x^{25}+x^{20}+2x^{19}+3x^{18} \\ &\quad +4x^{17}+2x^{16}+2x^{11}+4x^{10}+3x^9 \\ &\quad +2x^8+x^7+x^2+2x+1 \end{aligned}$$

Hence his Banzhaf index was

$$2\sum_{p=13}^{15} a_p = 0.$$

General Election, 11/04/2008

Banzhaf index of individual voter in Pennsylvania primary. Assume $N = 2K + 1$ voters.

$P_1(x) = (1+x)^{2K}$. Let B denote each voter's Banzhaf index.

$$B_1 = 2\begin{pmatrix} 2K \\ K \end{pmatrix} = 2\frac{(2K)!}{(K!)^2}$$

Use Stirling's formula:

$$R! \approx \sqrt{2\,R\,\pi}\left(\frac{R}{e}\right)^R.$$

The square root rule

$$B_1 = 2 \binom{2K}{K} \approx \frac{2^{2K+1}}{\sqrt{\pi K}} = \sqrt{\frac{2}{\pi N}} 2^N$$

Dividing by the number of voting combinations, 2^N, we find an individual voter's share is $\sqrt{2/(\pi N)}$.

Thus, to be sure that every vote has an even chance of making a critical difference, a representative's weight should be proportional to the square root of the population of the constituency.

This was Penrose's argument, rediscovered by Banzhaf.

Electoral College—Penrose style

There are 535 electors. Eight states have the minimum number (3), and the maximum is California with 55.

If we allocated electoral votes by the square root rule even the least populous state (Wyoming) would have 4 votes; California would have 29.

Banzhaf has advocated this change in a paper, “One Man, 3.312 Votes: A Mathematical Analysis of the Electoral College”, Villanova Law Review **16** (1968), 304–32.

Statistical facts

The Penrose-Banzhaf model is based on an assumption that all voting combinations are equally likely.

Do voters use a coin toss to decide which candidate to support?

This would involve two assumptions:

- Each candidate has probability 0.5 of receiving a given voter's support.
- Votes are independent.

Neither assumption is intuitively justified so one has to refer to previous elections.

A little history

- In 1984 the Mondale-Ferraro ticket carried just two states (MN and DC)
- In 2004, Utah delivered to Bush-Cheney 2.7 votes for every vote for Kerry-Lieberaman, and DC delivered 10 votes for Kerry-Lieberman for every Bush-Cheney vote.

Analysis of voting patterns over the years has shown that the hypothesis on which the square root rule is based is extremely unlikely.

Banzhaf's other idea

Banzhaf published another paper, "Weighted Voting Doesn't Work: A Mathematical Analysis", Rutgers Law Review, **19** (1965), 317–43.

There isn't really a fair way to arrange a two-tiered voting system. Democrats in Utah, and Republicans in DC are effectively disenfranchised.

If your state is red or blue, might as well go fishing. The only votes that count are in the swing states.

Perhaps the United States should consider direct election.